1 Section 8.3 Hyperbolas

A **hyperbola** is the set of all points in the plane, the difference between whose distances from two fixed points (**Foci**) is a positive constant.

The **transverse** axis is the line segment joining the intercepts. The midpoint of the transverse axis is called the **center** of the hyperbola. The **conjugate axis** passes through the center of the hyperbola and is perpendicular to the transverse axis. The **vertices** of a hyperbola are the points where the hyperbola intersects the transverse axis.

The **standard form** of the equation of a hyperbola with the center at the origin and transverse axis on the x-axis is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The coordinates of the vertices are $(a, 0)$ and $(-a, 0)$, and the coordinates of the foci are $(c, 0)$ and $(-c, 0)$, where $c^2 = a^2 + b^2$. The asymptotes are given by the equations $y = \frac{b}{a}x$ and $y = \frac{-b}{a}x$.

The **standard form** of the equation of a hyperbola with the center at the origin and transverse axis on the y-axis is given by $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. The coordinates of the vertices are $(0, a)$ and $(0, -a)$, and the coordinates of the foci are $(0, c)$ and $(0, -c)$, where $c^2 = a^2 + b^2$. The asymptotes are given by the equations $y = \frac{a}{b}x$ and $y = \frac{-a}{b}x$.

When the x^2 term is positive, the transverse axis is on the x-axis. However, when the y^2 term is positive, the transverse axis is on the y-axis.

Example 1 *Find the vertices, foci, and asymptotes of the hyperbola given by the equation* $\frac{x^2}{9} - \frac{y^2}{16} = 1$. *Sketch the graph.*

Standard Forms of the Equation of a Hyperbola with Center at (h, k)

Transverse Axis Parallel to the x-axis

The standard form of the equation of a hyperbola with center at (h, k) and transverse axis parallel to the x-axis is given by $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. The coordinates of the vertices are $V_1\left(h + a, k\right)$ and $V_2\left(h - a, k\right)$. The coordinates of the foci are $F_1\left(h + c, k\right)$ and $F_2\left(h - c, k\right)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are $y - k = \pm\frac{b}{a}\left(x - h\right)$.

Transverse Axis Parallel to the y-axis

The standard form of the equation of a hyperbola with center at (h, k) and transverse axis parallel to the y-axis is given by $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. The coordinates of the vertices are $V_1\left(h, k + a\right)$ and $V_2\left(h, k - a\right)$. The coordinates of the foci are $F_1\left(h, k + c\right)$ and $F_2\left(h, k - c\right)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are $y - k = \pm\frac{a}{b}\left(x - h\right)$.

Example 2 *Find the center, vertices, foci, and asymptotes given by the equation* $x^2 - y^2 - 4x - 4y + 4 = 0$. *Sketch the graph.*

Eccentricity of a Hyperbola

The eccentricity e of a hyperbola is the ratio of c to a, where c is the distance from the center to a focus and a is the length of the semi transverse axis. That is $e = \frac{c}{a}$. The eccentricity of a hyperbola is a measure of its "wideness."

For a hyperbola, $c > a$, and therefore $e > 1$. As the eccentricity of the hyperbola increases, the graph becomes wider and wider.

Example 3 *Find the equation in standard form of the hyperbola that satisfies the following conditions:*

1. Asymptotes $y = \frac{1}{2}x$ and $y = \frac{-1}{2}x$, vertices $(0, 4)$ and $(0, -4)$.
2. Vertices $(-1, 5)$ and $(-1, -1)$, foci $(-1, 7)$ and $(-1, -3)$.
3. Foci $(1, -2)$ and $(7, -2)$, slope of an asymptote is $\frac{5}{4}$.
4. Passing through $(4, 4)$, slope of an asymptote $\frac{1}{2}$, center $(7, 2)$, and transverse axis parallel to the y-axis.
5. Eccentricity $\frac{5}{2}$, center at the origin, and a focus at $(0, 10)$.

Example 4 *Use the eccentricity to find the equation in standard form of each hyperbola: 1) Vertices* $(2, 3)$ *and* $(-2, 3)$, *and eccentricity* $\frac{5}{2}$. *2) Center* $(-3, -3)$, *conjugate axis of length* 6, *and eccentricity* 2.

Example 5 *Use the definition of a hyperbola to find the equation of the hyperbola in standard form which has the following: foci* $(5, 0)$ *and* $(-5, 0)$ *and passes through the point* $\left(5, \frac{9}{4}\right)$.