1 Section 7.3 Vectors

Scalar quantities have a magnitude (numerical and unit desription) only like area, distance and speed. **Vector quantities** have a magnitude and a direction like force and velocity

A **vector** is a directed line segment. The length of the line segment is the magnitude of the vector, and the direction of the vector is measured by an angle.

The point A is called the initial point (or tail) of the vector and the point B is called the terminal point (or head) of the vector. The vector is denoted by $\overrightarrow{AB}$ and the magnitude is denoted by $||AB||$.

Equivalent vectors have the same magnitude and the same direction.

Multiplying a vector by a positive real number (other than 1) changes the magnitude of the vector but not its direction. Multiplying a vector by a negative real number a reverses the direction of the vector and multiplies the magnitude of the vector by $|a|$.

Vectors can be added graphically by using the **parallelogram method** or the **triangle method.**

In the triangle method, the tail of the vector U is placed at the head of the other vector V. In the parallelogram method, the tails of the two vectors U and V are placed together.

$V - U = V + (-U)$.

Vectors in a coordinate Plane

A vector can be moved in the plane as long as the magnitude and the direction are not changed. If $P_1\,(x_1, y_1)$ is the initial point of a vector and $P_2\,(x_2, y_2)$ is the terminal point, then an equivalent vector OP has its initial point at the origin and its terminal point at $P\,(a, b)$ where $a = x_2 - x_1$ and $b = y_2 - x_1$. The vector can be denoted by $v = \langle a, b\rangle$; a and b are called the components of the vector.

Example 1 *Find the components of a vector CD whose tail is the point $C\,(2, 5)$ and whose head is the point $D\,(3, -1)$. Determine a vector w that is equivalent to $CD$and has its initial point at the origin.*

The **magnitude** of the vector $v = \langle a, b\rangle$ is $||V|| = \sqrt{a^2 + b^2}$. The **direction angle** is the angle between the vector and the positive x-axis and we can find it by $\tan\alpha = \frac{b}{a}$.

Fundamental Vector Operations

If $v = \langle a, b\rangle$ and $w = \langle c, d\rangle$ are two vectors and k is a real number, then

1. $||v|| = \sqrt{a^2 + b^2}$
2. $v + w = \langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$
3. $kv = k\langle a, b\rangle = \langle ka, kb\rangle$

Example 2 *Given $v = \langle 12, -5\rangle$ and $w = \langle 2, 7\rangle$, find* 1)$||v||$ 2)$v+w$ 3)$-5v$ 4)$||3v - 4w||$

A **unit vector** is a vector whose magnitude is 1.

Example 3 *Which of the following is a unit vector?* 1)$v = \langle 1, 1 \rangle$ 2)$v = \langle 0, 1 \rangle$ 3)$v = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

A unit vector in the direction of a vector v is $\frac{v}{||v||}$.

Example 4 *Find a unit vector u in the direction of v=$\langle 2, -9 \rangle$.*

Definition of Unit Vectors i and j
$i = \langle 1, 0 \rangle$ $j = \langle 0, 1 \rangle$
If v is a vector and $v = \langle a_1, a_2 \rangle$, then $v = a_1 i + a_2 j$.

Example 5 *If $v = \langle 3, 4 \rangle$, represent v in terms of the unit vectors i and j.*

Example 6 *Given $v = 4i + 3j$ and $w = 6i - 3j$, find $4v + 5w$.*

Horizontal and Vertical Components of a Vector
Let $v = \langle a_1, a_2 \rangle$, where $v \neq 0$, the zero vector. Then $a_1 = ||v|| \cos \theta$ and $a_2 = ||v|| \sin \theta$ where θ is the angle between the positive x-axis and v.

The Horizontal component of v is $||v|| \cos \theta$. The Vertical component of v is $||v|| \sin \theta$.

Example 7 *Is $u = \cos \theta i + \sin \theta j$ a unit vector?*

Example 8 *Find the horizontal and vertical components of a vector v of magnitude 10 meters with direction angle $225°$. Write the vector in the form $v = a_1 i + a_2 j$.*

Definition of Dot Product
Given $v = \langle a, b \rangle$ and $w = \langle c, d \rangle$, the dot product of v and w is given by $v \cdot w = ac + bd$.

Example 9 *Find the dot product of $v = \langle 6, -2 \rangle$ and $w = \langle -2, 4 \rangle$.*

In the following properties, u, v, and w are vectors and a is a scalar.
1)$v \cdot w = w \cdot v$ 2)$u \cdot (v + w) = u \cdot v + u \cdot w$ 3)$a\,(u.v) = (au) \cdot v = u \cdot (av)$
4)$v \cdot v = ||v||^2$ 5)$0 \cdot v = 0$ 6)$i \cdot i = j \cdot j = 1$ 7)$i \cdot j = j \cdot i = 0$

Magnitude of a Vector in terms of the Dot Product
If $v = \langle a, b \rangle$, then $||v|| = \sqrt{v \cdot v}$

Alternative Formula for the Dot Product
If v and w are two nonzero vectors and α is the smallest non-negative angle between v and w, then $v \cdot w = ||v|| ||w|| \cos \alpha$.

Angle between Two Vectors
If v and w are two nonzero vectors and α is the smallest non-negative angle between v and w, then $\cos \alpha = \frac{v \cdot w}{||v|| ||w||}$ and $\alpha = \cos^{-1} \left(\frac{v \cdot w}{||v|| ||w||} \right)$.

Example 10 *Find the measure of the smallest positive angle between the vectors $v = 3i + 2j$ and $w = -2i - j$.*

Parallel and Perpendicular Vectors

Two vectors are **parallel** when the angle α between the vectors is $0°$or $180°$.

Two nonzero vectors v and w are **orthogonal (perpendicular)** if and only if $v \cdot w = 0$.

Scalar Projection

If v and w are two nonzero vectors and α is the smallest non-negative angle between v and w, then the scalar projection of v on w, $\text{proj}_w v$, is given by $\text{proj}_w v = ||v|| \cos\alpha$.

Example 11 *Given $v = 3i - 4j$ and $w = i + 3j$, find $proj_w v$.*

Example 12 *Find a vector that has the initial point $(3, -1)$ and is equivalent to $v = 2i - 3j$.*

Example 13 *Let $w = 4i + j$. Find a vector perpendicular to w.*