1 Section 6.5 Inverse Trigonometric Functions

The graph of $y = \sin x$ is not a one-to-one function and so has no inverse. If the domain of $y = \sin x$ is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then the graph of $y = \sin x$ has an inverse.

$y = \sin^{-1} x$ if and only if $x = \sin y$ where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Example 1 *If $y = \sin^{-1}\left(\frac{1}{2}\right)$, then y is the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where sine is $\frac{1}{2}$. Thus $y = \frac{\pi}{6}$.*

The graph of $y = \cos x$ is not a one-to-one function and so has no inverse. If the domain of $y = \cos x$ is restricted to $0 \le x \le \pi$, then the graph of $y = \cos x$ has an inverse.

$y = \cos^{-1} x$ if and only if $x = \cos y$ where $-1 \le x \le 1$ and $0 \le y \le \pi$.

See **Table 6.2 on page 506** in the book to see the graphs and domains and ranges for all of the remaining inverse trigonometric functions $y = \tan^{-1} x$, $y = \csc^{-1} x$, $y = \sec^{-1} x$, and $y = \cot^{-1} x$.

Example 2 *Find the exact value of each inverse function:* 1) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ 2) $\cot^{-1}\sqrt{3}$.

Example 3 *Prove the following: 1) If $x \le -1$ or $x \ge 1$, then $\csc^{-1} x = \sin^{-1}\frac{1}{x}$ and $\sec^{-1} x = \cos^{-1}\frac{1}{x}$.*

$$2)\quad \cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x}, & \text{for } x > 0. \\ \tan^{-1}\frac{1}{x} + \pi, & \text{for } x < 0 \\ \frac{\pi}{2}, & \text{for } x = 0 \end{cases}$$

Composition of Trigonometric Functions and Their Inverses

If $-1 \le x \le 1$, then $\sin(\sin^{-1} x) = x$, and $\cos(\cos^{-1} x) = x$.

If x is any real number, then $\tan(\tan^{-1} x) = x$.

If $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then $\sin^{-1}(\sin x) = x$.

If $-0 \le x \le \pi$, then $\cos^{-1}(\cos x) = x$.

If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\tan^{-1}(\tan x) = x$.

Example 4 *Find the exact value of each composition of functions:*

1) $\cos(\cos^{-1} 0.25)$ 2) $\cos^{-1}(\cos 0.25)$ 3) $\sin^{-1}(\sin 1)$ 4) $\sin(\sin^{-1} 2)$ 5) $\tan(\tan^{-1}\pi)$ 6) $\tan^{-1}(\tan\pi)$.

Example 5 *In which quadrant is each of the following:*

1) $\sin^{-1}\frac{1}{3}$ 2) $\sin^{-1}\left(-\frac{1}{3}\right)$ 3) $\cos^{-1}\frac{1}{3}$ 4) $\cos^{-1}\left(-\frac{1}{3}\right)$ 5) $\tan^{-1}\frac{1}{3}$ 6) $\tan^{-1}\left(-\frac{1}{3}\right)$

Example 6 *Find the exact value of the following:* 1) $\cos\left(\sin^{-1}\left(-\frac{5}{6}\right)\right)$ 2) $\cos(\sin^{-1}\left(\frac{4}{5}\right) - \cos^{-1}\left(\frac{5}{17}\right))$

Example 7 *Solve for x in the following:* 1) $\cos^{-1} x = \alpha + \beta$ 2) $\sin^{-1} x = \alpha + \beta$.

Example 8 *Solve the trigonometric equation* $\cos^{-1}\left(\frac{5}{13}\right) + \sin^{-1} x = \frac{\pi}{2}$.

Example 9 *Verify the identity:* $\cos^{-1} x + \cos^{-1}(-x) = \pi$.

Graphs of the Inverse Trigonometric Functions

Example 10 *Graph* $y = \sin^{-1}(x+1) - 2$

Example 11 *Find the exact value of:*

1) $\tan^{-1}\left(\sin \frac{\pi}{6}\right)$ 2) $\tan\left(2\sin^{-1} \frac{\sqrt{3}}{2}\right)$ 3) $\cos\left(\sin^{-1} \frac{3}{4} + \cos^{-1} \frac{5}{13}\right)$

Example 12 *Evaluate the expression* $y = \sec\left(\sin^{-1} x\right)$.