

# 1 Section 5.4 Trigonometric Functions of Real Numbers

## The Wrapping Function

A circle given by the equation  $x^2 + y^2 = 1$  is called a unit circle and let us assume that we have a vertical coordinate line  $l$  tangent to the unit circle at  $(1, 0)$ . We define a function  $W$  that pairs a real number  $t$  on the coordinate line with a point  $P(x, y)$  on the unit circle. This function is called the wrapping function because it is analogous to wrapping a line around a circle.

The positive part of the coordinate line is wrapped around the unit circle in a counterclockwise direction. The negative part of the coordinate line is wrapped around the circle in a clockwise direction. The wrapping function is defined by the equation  $W(t) = P(x, y)$ , where  $t$  is a real number and  $P(x, y)$  is the point on the unit circle that corresponds to  $t$ . On a unit circle, the measure of a central angle and the length of its arc can be represented by the same real number  $t$ .

**Example 1** Find the values of  $x$  and  $y$  such that  $W\left(\frac{\pi}{4}\right) = P(x, y)$ .

Let  $W$  be the **wrapping function**,  $t$  be a real number, and  $W(t) = P(x, y)$ . Then

$$\sin t = y \quad \cos t = x \quad \tan t = \frac{y}{x}, x \neq 0 \quad \csc t = \frac{1}{y}, y \neq 0 \quad \sec t = \frac{1}{x}, x \neq 0 \quad \cot t = \frac{x}{y}, y \neq 0.$$

Trigonometric functions of real numbers are frequently called **circular functions**. Note that the value of a trigonometric function at the real number  $t$  is its value at an angle of  $t$  radians. There is a difference between the definition in section 5.3 and section 5.4 in which the domains of the circular functions (section 5.4) are sets of real numbers while the domains are sets of angles in section 5.3.

**Example 2** Answer True or False: 1) the real number  $\pi$  is the same as the angle with a measure of  $\pi$  radians. 2)  $\sin \pi = \sin(\pi \text{ radians})$

**Example 3** Find the exact value of each function:

$$1) \sin \frac{\pi}{6} \quad 2) \cos \left(\frac{-11\pi}{6}\right) \quad 3) \tan \left(\frac{-5\pi}{3}\right) \quad 4) \csc \frac{7\pi}{6}$$

**Domain and Range of the Trigonometric Functions** ( $n$  is an integer)

Function	Domain	Range
$y = \sin t$	$\{t \mid -\infty < t < \infty\}$	$\{y \mid -1 \leq y \leq 1\}$
$y = \cos t$	$\{t \mid -\infty < t < \infty\}$	$\{y \mid -1 \leq y \leq 1\}$
$y = \tan t$	$\{t \mid -\infty < t < \infty, t \neq (2n+1)\pi/2\}$	$\{y \mid -\infty < y < \infty\}$
$y = \sec t$	$\{t \mid -\infty < t < \infty, t \neq n\pi\}$	$\{y \mid y \geq 1, y \leq -1\}$
$y = \csc t$	$\{t \mid -\infty < t < \infty, t \neq (2n+1)\pi/2\}$	$\{y \mid y \geq 1, y \leq -1\}$
$y = \cot t$	$\{t \mid -\infty < t < \infty, t \neq n\pi\}$	$\{y \mid -\infty < y < \infty\}$

If we consider the points  $t$  and  $-t$  on the coordinate line  $l$  tangent to the unit circle at the point  $(1, 0)$ , then we will get that for all  $t$  in the domains

$\sin(-t) = -\sin t$ ,  $\cos(-t) = \cos t$ ,  $\tan(-t) = -\tan t$ ,  $\csc(-t) = -\csc t$ ,  
 $\sec(-t) = \sec t$ , and  $\cot(-t) = -\cot t$ .

The **odd trigonometric functions** are  $y = \sin t$ ,  $y = \csc t$ ,  $y = \tan t$ , and  $y = \cot t$ . The **even trigonometric functions** are  $y = \cos t$  and  $y = \sec t$ .

**Example 4** Is the function defined by  $g(x) = x^2 - \sec x$  even, odd, or neither?

The **period** of a function is the smallest value of  $p$  for which  $f(t) = f(t+p)$ .

Let  $W$  be the wrapping function,  $t$  be a point on the coordinate line tangent to the unit circle at  $(1, 0)$  and  $W(t) = P(x, y)$ . Thus, the value of the wrapping function repeats itself in  $2\pi$  units. The wrapping function is periodic and the period is  $2\pi$ . Also, we get that  $\cos(t+2\pi) = \cos t$ ,  $\sin(t+2\pi) = \sin t$ ,  $\sec(t+2\pi) = \sec t$ , and  $\csc(t+2\pi) = \csc t$ .

The **period** of  $\cos t$ ,  $\sin t$ ,  $\sec t$ ,  $\csc t$  is  $2\pi$ .

By using the same idea, and if  $W(t) = P(x, y)$ , then  $W(t+\pi) = P(-x, -y)$ . That means  $\tan(t+\pi) = \tan t$  and  $\cot(t+\pi) = \cot t$ .

The **period** of  $\tan t$  and  $\cot t$  is  $\pi$ .

#### Fundamental Trigonometric Identities

The **reciprocal identities** are  $\sin t = \frac{1}{\csc t}$      $\cos t = \frac{1}{\sec t}$      $\tan t = \frac{1}{\cot t}$

The **ratio identities** are  $\tan t = \frac{\sin t}{\cos t}$      $\cot t = \frac{\cos t}{\sin t}$

The **Pythagorean identities** are  $\cos^2 t + \sin^2 t = 1$      $1 + \tan^2 t = \sec^2 t$      $1 + \cot^2 t = \csc^2 t$

**Example 5** By using the unit circle and the definitions of the trigonometric functions, show that  $\cos(t+\pi) = -\cos t$ .

**Example 6** Write the expression  $\sin^2 \theta + \cos^2 \theta + \tan^2 \theta$  as a single term.

**Example 7** For  $\pi < t < \frac{3\pi}{2}$ , write  $\tan t$  in terms of  $\cos t$ .

**Example 8** Perform the indicated operation and simplify: 1)  $(1 - \sin t)(1 + \sin t)$     2)  $\frac{\sin t}{1 + \cos t} + \frac{1 + \cos t}{\sin t}$

**Example 9** Given  $\sin t = \frac{1}{2}$ ,  $\frac{\pi}{2} < t < \pi$ , find  $\tan t$ .