

# 1 Section 5.3 Trigonometric Functions of Any Angle

Let  $P(x, y)$  be any point, except the origin, on the terminal side of an angle  $\theta$  in standard position. Let  $r = d(O, P)$ , the distance from the origin to  $P$ . The six trigonometric functions of  $\theta$  are

$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \tan \theta = \frac{y}{x}, x \neq 0 \quad \csc \theta = \frac{r}{y}, y \neq 0 \quad \sec \theta = \frac{r}{x}, x \neq 0 \quad \cot \theta = \frac{x}{y}, y \neq 0 \text{ where } r = \sqrt{x^2 + y^2}.$$

Note that the value of any trigonometric function is independent of the point chosen on the terminal side of the angle.

**Example 1** Find the exact value of the six trigonometric functions of an angle in standard position whose terminal side contains the point  $(4, -1)$ .

## Values of Trigonometric Functions for Quadrantal Angles

The terminal side of  $0^\circ$  coincides with the positive x-axis. Let  $P(x, 0)$ ,  $x > 0$ , be any point on the x-axis. Then  $y = 0$ , and  $r = x$ . The value of the six trigonometric functions of  $0^\circ$  are  $\sin 0^\circ = \frac{0}{r} = 0$ ,  $\cos 0^\circ = 1$ ,  $\tan 0^\circ = 0$ ,  $\csc 0^\circ$  is undefined,  $\sec 0^\circ = 1$ , and  $\cot 0^\circ$  is undefined.

### Values of Trigonometric Functions for Quadrantal Angles

$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
$0^\circ$	0	1	0	undefined	1	undefined
$90^\circ$	1	0	undefined	1	undefined	0
$180^\circ$	0	-1	0	undefined	-1	undefined
$270^\circ$	-1	0	undefined	-1	undefined	0

The sign of a trigonometric function depends on the quadrant in which the terminal side of the angle lies. For example, if  $\theta$  is an angle whose terminal side lies in Quadrant *II* and  $P(x, y)$  is on the terminal side of  $\theta$ , then both  $x$  and  $y$  are negative.

### Signs of the Trigonometric Functions

sign of	I	II	III	IV
$\sin \theta$ and $\csc \theta$	positive	positive	negative	negative
$\cos \theta$ and $\sec \theta$	positive	negative	negative	positive
$\tan \theta$ and $\cot \theta$	positive	negative	positive	negative

**Example 2** Determine each of the following: 1)  $\sin 0^\circ$     2)  $\cos 90^\circ$     3)  $\sin 180^\circ$     4)  $\cos 270^\circ$ .

**Example 3** True or False: 1) If  $\sin \theta$  is positive, then  $\theta$  must be in Quadrant *I*.

2) If  $\sin \theta$  is negative, then  $\theta$  must be in either Quadrant *III* or Quadrant *IV*.

3) If  $\sin \theta$  is negative and  $\cos \theta$  is positive, then  $\theta$  must be in Quadrant *IV*.

Given  $\sec \theta = 3$  and  $\sin \theta < 0$ , find  $\tan \theta$  and  $\csc \theta$ .

**Reference Angle**

Given  $\angle \theta$  in standard position, its reference angle  $\theta'$  is the smallest positive angle formed by the terminal side of  $\angle \theta$  and the x-axis.

**Example 4** For each of the following angles  $\theta$  determine the measures of its reference angle  $\theta'$ . 1)  $\theta = 135^\circ$     2)  $\theta = 330^\circ$     3)  $\theta = \frac{10\pi}{3}$     4)  $\theta = -6$     5)  $\theta = 30$

**Reference Angle Theorem**

To evaluate  $\sin \theta$ , determine  $\sin \theta'$ . Then use either  $\sin \theta'$  or its opposite as the answer, depending on which has the correct sign.

**Example 5** True or False: 1)  $\sin 150^\circ = \sin 30^\circ$     2)  $\cos 150^\circ = \cos 30^\circ$     3)  $\tan 150^\circ = \tan 30^\circ$     4)  $\sin(-45^\circ) = -\sin 45^\circ$     5)  $\cos(-45^\circ) = \cos 45^\circ$

**Example 6** Determine the exact value of each function

1)  $\sin 330^\circ$     2)  $\cos 660^\circ$     3)  $\tan 240^\circ$     4)  $\csc(-510^\circ)$

**Example 7** Find the exact value of each expression 1)  $\tan 225^\circ + \sin 240^\circ \cos 60^\circ$     2)  $\sin^2\left(\frac{5\pi}{4}\right) + \cos^2\left(\frac{5\pi}{4}\right)$