1 Section 3.5 Rational Functions and Their Graphs

$F(x) = \frac{P(x)}{Q(x)}$ is called a **rational function** where $P(x)$and $Q(x)$ are polynomials. The **Domain** of F is the set of all real numbers except those for which $Q(x) = 0$.

Example 1 *Find the Domain of* $F(x) = \frac{x^2-x-5}{x(2x-5)(x+3)}$.

The graph of $G(x) = \frac{x+1}{x-2}$ is given in the figure. The graph has the following properties:

1) The graph has an x-intercept at $(-1, 0)$ and a y-intercept at $\left(0, -\frac{1}{2}\right)$.

2) The graph does not exist when $x = 2$.

3) As x takes on values that are close to 2 but less than 2, i.e., "As x approaches 2 from the left", The function values $G(x)$ decrease without bound, i.e., "$G(x)$ approaches negative infinity".

x	1.9	1.95	1.99	1.995	1.999
$G(x)$	-29	-59	-299	-599	$-2,999$

$G(x) \longrightarrow -\infty$ as $x \longrightarrow 2^-$

4) As $x$approaches 2 from the right, $G(x)$approaches positive infinity".

Definition of a Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function F provided that

$F(x) \longrightarrow \infty$ or $F(x) \longrightarrow -\infty$ as x approaches a from either left or right.

$x = 2$ is a vertical asymptote of the graph of $G(x)$. (See 3) and 4) above)

5) As x increases without bound, the values of $G(x)$ are becoming closer to 1.

x	1000	5000	10,000	50,000	100,000
$G(x)$	1.0030	1.0060	1.00030	1.00060	1.00003

$G(x) \longrightarrow 1$ as $x \longrightarrow \infty$.

6) As x decrease without bound, the values of $G(x)$ are becoming closer to 1.

$G(x) \longrightarrow 1$ as $x \longrightarrow -\infty$.

Definition of a Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function F provided that

$F(x) \longrightarrow b$ as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$.

$y = 1$ is a horizontal asymptote of the graph of $G(x)$. (See 5) and 6) above)

We note the following:

1) A rational function may have several vertical asymptotes, but it can have at most one horizontal asymptote.

2) The graph may intersect its horizontal asymptote.

3) The graph does not intersect its vertical asymptote.

Theorem on Vertical Asymptote

If the real number a is a zero of the denominator $Q(x)$, then the graph of $F(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ have no common factors, has the vertical asymptote $x = a$.

Example 2 *Find the vertical asymptote of each rational function:*

1)$F(x) = \frac{x^2-4}{x^2-3x-4}$ 2)$G(x) = \frac{x^3}{x^2+1}$ 3)$H(x) = \frac{x^2-3x-4}{x^2-6x+8}$

Theorem on Horizontal Asymptotes

Let $F(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$ be a rational function with numerator of degree n and denominator of degree m.

1) If $n < m$, then the x-axis ($y = 0$) is the horizontal asymptote of the Graph of F.

2) If $n = m$, then the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the Graph of F.

3) If $n > m$, then the Graph of F has no horizontal asymptote.

Example 3 *Find the Horizontal Asymptote of each rational function.*

1)$F(x) = \frac{x^2-9}{x^3-36x}$ 2)$G(x) = \frac{3x^2-9}{x-36}$ 3)$H(x) = \frac{6x^2-9}{2x^2+5x+2}$

We note that the Zeros and Vertical asymptotes of a rational function F divide the x-axis into intervals where in each interval

$F(x)$ is positive for all x in the interval or $F(x)$ is positive for all x in the interval.

General Procedure for Graphing Rational Functions that Have no Common Factors

Assume $F(x) = \frac{P(x)}{Q(x)}$.

1. Find the Asymptotes
   (a) Vertical Asymptotes: Solve $G(x) = 0$.
   (b) Horizontal Asymptote: Apply the theorem on the horizontal asymptotes.
2. Find the x- and y-intercepts of $F(x)$
   (a) x-intercepts: Solve $P(x) = 0$.
   (b) y-intercept: Find $F(0)$.
3. Check Symmetry with respect to the x-axis, the y-axis, or the origin.
4. Divide the x-axis into intervals by using the vertical asymptotes and the x-intercepts and take a point in each interval.
5. Determine the behavior near asymptotes.
6. Determine whether the graph of $F(x)$intersects its horizontal asymptote at any point.
7. Complete the Sketch.

Example 4 *Sketch the graph of the following rational functions:*

1)$F(x) = \frac{2x^2-18}{x^2+3}$ 2)$G(x) = \frac{x^2+x-12}{x^2+x-6}$

Slant Asymptotes

The rational function given by $F(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ have no common factors, has a slant asymptote if the degree of the polynomial $P(x)$ in the numerator is one greater than the degree of the polynomial $Q(x)$ in the denominator and to find it, we divide $P(x)$ by $Q(x)$, i.e., $F(x) = \frac{P(x)}{Q(x)} = (mx+b) + \frac{r(x)}{Q(x)}$ where the degree of $r(x)$ is less than the degree of $Q(x)$.The line $y = mx + b$ is the slant asymptote.

Example 5 *Find the slant asymptote of* $F(x) = \frac{4x^3+7x^2+22x-8}{x^2+2x+5}$

Example 6 *Sketch the graph of* $G(x) = \frac{4x^2-9}{x+2}$

Graph a rational function that has a common factor

1) Reduce the rational function to lowest terms.
2) Sketch the graph of the new function using the general procedure.
3) Find the coordinates of any holes in the function.

Example 7 *Sketch the graph of* $G(x) = \frac{x^2-x-2}{x^2+x-6}$

Does $G(x)$ have a vertical asymptote when $x = 2$?
Find the coordinates of any holes in $G(x)$.

Example 8 ***Example 9*** *Determine the two points where the graph of* $F(x) = \frac{x^3+x^2+4x+1}{x^3+1}$ *intersects its horizontal asymptote.*

Example 10 *Determine the point where the graph of* $F(x) = \frac{3x^3+2x^2-8x-12}{x^2+4}$ *intersects its slant asymptote.*