1 Section 3.4 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $P(x)$ has at least one complex zero.

The Number of Zeros of a Polynomial

If $P(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $P(x)$ has exactly n complex zeros, provided that each zero is counted according to its multiplicity.

Example 1 *How many complex zeros does* $P(x) = x^4 - 9x^3 - 58x^2 + 159x - 36$ *have, provided that each zero is counted according to its multiplicity.*

The Conjugate Pair Theorem

If $a + bi$ $(b \neq 0)$ is a complex zero of the polynomial $P(z)$, with real coefficients, then the conjugate $a - bi$ is also a complex zero of the polynomial.

Example 2 *TRUE or FALSE Complex zeros always occur in Pairs. (Example* $(x - i)$*)*

Example 3 *If* $3-2i$ *is a complex zero of* $x^2-6x+13$*, what is the other complex zero?*

Example 4 *Find all zeros of* $x^4 - 4x^3 + 14x^2 - 36x + 45$ *given that* $2 + i$ *is a zero. (Used two methods)*

Example 5 *Does the graph of* $y = x^4 - 4x^3 + 14x^2 - 36x + 45$ *intersect the x-axis?*

Example 6 *Find all zeros of* $x^5 - 4x^4 + 14x^3 + 144x^2 + 45x + 820$*,given that* $4 + 5i$ *is a zero.*

Example 7 *Solve:* $x^4 - 14x^3 + 20x^2 - 150x + 125 = 0$*, given that* $(5, 0)$ *is an x-intercept and the only x-intercept.*

Linear and Quadratic Factors of a Polynomial

Every polynomial with real coefficients and positive degree n can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Example 8 *Write* $x^4 - 1$ *as a product of linear factors and quadratic factors that are irreducible over the reals.*

Example 9 *Write each polynomial as a product of linear factors and quadratic factors that are irreducible over the reals.*

1)$P(x) = 2x^3 - 5x^2 + 8x - 20$ 2)$P(x) = x^3 + x^2 + x - 14$.

Example 10 *Find each Polynomial:*

1) A polynomial of lowest degree that has $\frac{1}{2}$, $4+i$, $4-i$, as zeros.
2) A polynomial of degree 3 that has 1, -1, and 4 as zeros.
3) A polynomial of degree 4 that has real coefficients and zeros $-3i$ and $2+5i$.
4) A polynomial of degree 3 that has real coefficients and zeros 0 and $3+i$.
5) The polynomial $P(x)$ of degree 5 that has 1 as a zero of multiplicity 2 and 2 as a zero of multiplicity 3 and $P(-1)=-54$.

A polynomial that has a given set of zeros is not unique. For example, x^3-7x+6 and $2x^3-14x+12$ have zeros 1, 2 and -3. The graphs of the two polynomials are different, but they have the same x-intercepts.

Example 11 *Is it possible that the degree of the polynomial in the figure is* 3?

Example 12 *Does the graph of each polynomial of degree $n \geq 1$ have at least one x-intercept?*

Example 13 *Verify that $x^3-x^2-ix^2-9x+9i+9$ has $1+i$ as a zero but its conjugate $1-i$ is not a zero. Explain why this does not contradict the Conjugate Pair Theorem.*

Example 14 *Show that* 2 *is a zero of multiplicity* 3 *of $P(x)=x^5-6x^4+21x^3-62x^2+108x-72$ and express $P(x)$ as a product of linear factors and/or quadratic factors that are irreducible over the reals.*