1 Section 3.3 Zeros of Polynomial Functions

Definition of Multiple Zeros of a Polynomial

If a polynomial $P(x)$ has $(x-r)$ as a factor exactly k times, then r is a zero of multiplicity k of the polynomial $P(x)$.

Example 1 *Find the zeros of the following polynomial and state the multiplicity of each zero.* $P(x) = (x-5)^2 (x+2)^3 (x+4)$

A zero of multiplicity 1 is generally referred to as a simple zero.

Number of Zeros of a Polynomial Functions

A polynomial function P of degree n has at most n zeros, where each zero of multiplicity k is counted k times.

The Rational Zero Theorem

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, and $\frac{p}{q}$ (where p and q have no common factors) is a rational zero of $P(x)$, then p is a factor of a_0 and q is a factor of a_n.

Example 2 *Use the Rational Zero Theorem to list all possible rational zeros of* $P(x) = 6x^3 + 5x^2 - 11x + 12$.

A real number b is called an **upper bound** of the zeros of the polynomial function P if no zero is greater than b.

A real number a is called an **lower bound** of the zeros of the polynomial function P if no zero is less than a.

Upper- and Lower- Bound Theorem

Upper Bound If $b > 0$ and all the numbers in the bottom row of the synthetic division of P by $(x-b)$ are either positive or zero, then b is an upper bound for the real zeros of P.

Lower Bound If $a < 0$ and the numbers in the bottom row of the synthetic division of P by $(x-a)$ alternate in sign (the number zero can be considered positive or negative), then a is an lower bound for the real zeros of P.

Note that upper and lower bounds are not unique.

Example 3 *Find the smallest positive integer and the largest negative integer that , by the Upper- and Lower- Bound Theorem, are upper and lower bounds for the real zeros of* $P(x) = 2x^3 + 7x^2 - 4x - 14$.

The number of variations in sign of the coefficients of a polynomial $P(x)$ or $P(-x)$ refers to sign changes of the coefficients from positive to negative or from negative to positive that we find when we have successive terms of the polynomial. The terms of the polynomial are assumed to appear in the order of descending powers of x.

Example 4 *Determine the number of variations in sign of* $P(x)$ *and* $P(-x)$ *where* $P(x) = -3x^3 + 4x^2 - 2x - 10$.

Descartes' Rule of Signs

Let $P(x)$ be a polynomial with real coefficients and with the terms arranged in the order of decreasing powers of x.

1. The number of positive real zeros of $P(x)$ is equal to the number of variations in sign of $P(x)$ or is equal to that number decreased by an even integer.
2. The number of negative real zeros of $P(x)$ is equal to the number of variations in sign of $P(-x)$ or is equal to that number decreased by an even integer.

Example 5 *Determine both the number of possible positive and the number of possible negative real zeros of each polynomial:*

1)$P(x) = x^3 - x^2 - 2x + 3$ 2)$P(x) = -4x^5 + x^4 - 10x^3 + 5x + 4$.

In applying Descartes' Rule of Signs, we count each zero of multiplicity k as k zeros.

Find Zeros of a Polynomial

1. Gather general information.
   (a) Determine the degree n of the polynomial and so the number of zeros is at most n.
   (b) Apply Descartes' Rule of Signs to find the possible number of positive zeros and also the possible number of negative zeros.
2. Check Suspects.
   (a) Apply the Rational Zero Theorem to list rational numbers that are possible zeros.
   (b) Use synthetic division to test numbers in your list.
   (c) If you find an upper or a lower bound, then eliminate from your list any number that is greater than an upper bound or less than a lower bound.
3. Each time a zero is found, you obtain a reduced polynomial.
   (a) If a reduced polynomial is of degree 2, then find its zeros either by factoring or by applying the quadratic formula.
   (b) If the degree of a reduced polynomial is 3 or greater, then repeat the above steps for this polynomial.

Example 6 *Find the zeros of* 1)$P(x) = 3x^4+23x^3+56x^2+52x+16$. 2)$P(x) = 2x^5 - x^4 - 15x^3 + 19x^2 + x - 6$.

Example 7 *The dimensions of a rectangular box are consecutive even natural numbers. The volume of that box is* 960 *cubic inches. Find the dimensions of the box.*

Example 8 *Verify that* $x^6 + 3x^4 + 3x^2 + 1$ *has no rational zeros.*