1 2.6 Algebra of Functions

Operations on Functions

For all values of x for which $f(x)$ and $g(x)$ are defined, we define the following functions:

Sum $(f+g)(x) = f(x) + g(x)$

Difference $(f-g)(x) = f(x) - g(x)$

Product $(fg)(x) = f(x)\,g(x)$

Quotient $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \qquad g(x) \neq 0.$

Domain of $f+g,\ f-g,\ fg,\ \frac{f}{g}$

For the given functions f and g, the domains of $f+g$, $f-g$, and $f \cdot g$ consist of all real numbers formed by the intersection of the domains of f and g. The domain of $\frac{f}{g}$ is the set of all real numbers formed by the intersection of the domains of f and g, except for those real numbers x such that $g(x) = 0$.

Example 1 *Given* $f(x) = x^2$ *and* $g(x) = 2x$, *find*
1)$(f+g)(x)$ *2)*$(f-g)(x)$ *3)*$(fg)(x)$ *4)*$\left(\frac{f}{g}\right)(x)$

Example 2 *If* $f(x) = x^2 - 25$ *and* $g(x) = \sqrt{x+3}$, *then find the domain of* $f+g,\ f-g,\ fg,$ *and* $\frac{f}{g}$.

Example 3 *If* $f(x) = \sqrt{2-x}$ *and* $g(x) = \sqrt{x+3}$, *then find the domain of* $\left(\frac{f}{g}\right)(x)$

Example 4 *Let* $f(x) = x^2+2$ *and* $g(x) = 3x-1$. *Find* 1) $(f+g)(3)$ 2)$fg(4)$ 3) $\left(\frac{f}{g}\right)(-1)$

The Difference Quotient of f is the expression $\frac{f(x+h)-f(x)}{h}, \qquad h \neq 0.$

Example 5 *Determine the difference quotient of* $f(x) = 2x^2 + 5x - 3$.

Composition of Functions

For the functions f and g, the composite function or composition of f by g is given by $(g \circ f)(x) = g(f(x))$ for all x in the domain of f such that $f(x)$ is in the domain of g.

Example 6 *If* $f(x) = 2x^2+3x+1$ *and* $g(x) = 4x-5$, *then find 1)*$(g \circ f)(x)$ 2) $(f \circ g)(x)$

Note that in this example $(f \circ g) \neq (g \circ f)$

Example 7 *If* $f(x) = 2x - 1$ *and* $g(x) = x^3 - 3x$, *then find* $(g \circ f)(x)$.

Example 8 *If* $f(x) = 2x - 1$ *and* $(f \circ g)(x) = 2x + 1$, *then find* $g(x)$.

Example 9 *If* $(f \circ g)(x) = 10 - x,\ f(x) = 2x + 4$ *and* $g(x) = ax + b$, *where* a, *and* b *are real numbers, then find the values of* a *and* b.

Note that in this example $(f \circ g) \neq (g \circ f)$

To evaluate $(f \circ g)(c)$ for some constant c, you can use either of the following methods:

1) First evaluate $g(c)$ and then substitute this result for x in $f(x)$.

2)First determine $f(g(x))$ and then substitute c for x.

Example 10 *Evaluate* $(f \circ g)(-2)$, *where* $f(x) = x + 3$ *and* $g(x) = 2x^2 - 9$.

Example 11 *If* $f(x) = \sqrt{16 + \sqrt{x}}$, *then find the value of* $(f \circ f)(0)$.

Example 12 *If* $f(x) = \begin{cases} 2x - 1 & \text{if } x \leq -1 \\ 2x + 3 & \text{if } x > -1 \end{cases}$, *and* $g(x) = [x]$, *where* $[x]$ *is the greatest integer function, then find the value of* $(f \circ g)(-0.3) + \sqrt{(f \cdot g)(0.5)}$.

Example 13 *Given* $(g \circ f)(k) = 1$, *where* $f(x) = x + 1$ *and* $g(x) = 2 - x^2$, *then find the set of all possible values of* k.

Caution Sometimes, we need to adjust the domain of f so that $g(f(x))$ can always be evaluated. The idea is we want the range of f to be a part of the domain of g.

Example 14 *Let* $f(x) = 2x + 3$ *and* $g(x) = \frac{1}{x+1}$. *What value of* x *must be excluded from the domain of* f *so that* $g(f(x))$ *can always be evaluated?*

Example 15 *Let* $f(x) = x + 1$ *and* $g(x) = \sqrt{x - 4}$. *What value of* x *must be excluded from the domain of* f *so that* $g(f(x))$ *can always be evaluated?*

Example 16 *Let* $f(x) = x^2 - 12x + 36$ *and* $g(x) = \sqrt{-x}$, *then find* $(g \circ f)(x)$.