1 Section 10.4 Determinants

Associated with each square matrix A there is a number called the determinant of A. We will denote the determinant of the matrix A by $\det(A)$ or by $|A|$.

Determinant of a 2×2 Matrix

The determinant of the matrix $A = [a_{ij}]$ of order 2 is $|A| = a_{11}a_{22} - a_{21}a_{12}$.

Example 1 *True or False:* 1) $[A] = |A|$ 2)$|A|$ *is a number.*

Example 2 *Find the value of the determinant of the matrix* $A = \begin{bmatrix} 9 & 1 \\ 8 & 2 \end{bmatrix}$.

The Minor of a Matrix

The minor M_{ij} of the element a_{ij} of a square matrix A of order $n \geq 3$ is the determinant of the matrix of order $n-1$ obtained by deleting the ith row and the jth column of A.

Example 3 *Given the matrix* $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & -2 & 9 \\ 5 & -4 & 8 \end{bmatrix}$, *find* M_{13}.

Cofactor of a Matrix

The cofactor C_{ij} of the element a_{ij} of a square matrix A is given by $C_{ij} = (-1)^{i+j} M_{ij}$ is the minor of a_{ij}.

Example 4 *Given* $A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 4 & 0 \\ 2 & 1 & 1 \end{bmatrix}$, *find* M_{12}, C_{22} *and* C_{23}.

Determinants by Expanding by Cofactors

Given the square matrix A of order 3 or greater, the value of the determinant of A is the sum of the products of the elements of any row or column and their cofactors. For the rth row of A, the value of the determinant of A is $|A| = a_{r1}C_{r1} + a_{r2}C_{r2} + \cdots + a_{rn}C_{rn}$.

For the cth column of A, the determinant of A is $|A| = a_{1c}C_{1c} + a_{2c}C_{2c} + \cdots + a_{nc}C_{nc}$.

Example 5 *Evaluate the determinant of* $A = \begin{bmatrix} -2 & 3 & 1 \\ -3 & -4 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ *by expanding by cofactors.*

Effects of Elementary Row Operations on the Value of a Determinant of a Matrix

If A is a matrix of order n, then the following elementary row operations produce the indicated changes in the determinant of A.

1. Interchanging any two rows of A changes the sign of $|A|$.

2. Multiplying a row of A by a constant k multiplies the determinant of A by k.

3. Adding a multiple of a row of A to another row does not change the value of the determinant of A.

Example 6 *True or False:*

1. $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = -\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

2. $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 4 \\ 8 & 12 \end{vmatrix} = 4\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 8 \\ 12 & 16 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

5. $\begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix} = 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

6. $\begin{vmatrix} 4 & 8 \\ 12 & 16 \end{vmatrix} = 16\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix}$

These elementary row operations are often used to rewrite a matrix in triangular form. A matrix is in **triangular form** if all elements below or above the main diagonal are zero.

Example 7 $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 0 & -2 & 4 & 2 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 6 & 4 & -2 & 0 \\ 8 & 3 & 4 & 2 \end{bmatrix}$ *are in triangular form.*

Determinant of a Matrix in Triangular Form

Let A be a square matrix of order n in triangular form. The determinant of A is the product of the elements on the main diagonal.

$|A| = a_{11}a_{22}\cdots a_{nn}$.

Example 8 *Find* $|A|$*and* $|B|$ *from the pervious example.*

Example 9 *Evaluate the determinant by rewriting it in triangular form* $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 4 & 3 & 0 & 5 \\ -2 & -1 & 3 & -4 \\ 4 & 2 & 0 & 5 \end{bmatrix}$.

Conditions for a Zero Determinant

If A is a square matrix, then $|A| = 0$ when any one of the following is true.

1. A row (column) consists entirely of zeros.
2. Two rows (columns) are identical.
3. One row (column) is a constant multiple of a second row(column).

Example 10 *Tell which condition for being a zero determinant is met by each of the following determinants* 1) $\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 3 \end{vmatrix}$ 2) $\begin{vmatrix} 5 & 1 & 3 \\ 2 & 2 & 6 \\ 6 & 1 & 3 \end{vmatrix}$ 3) $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix}$

Product Property of Determinants

If A and B are square matrices of order n, then $|AB| = |A||B|$.

Existence of the Inverse of a Square Matrix

If A is a square matrix of order n, then A has a multiplicative inverse if and only if $|A| \neq 0$. Furthermore, $|A^{-1}| = \frac{1}{|A|}$.

Example 11 *Show that the determinant* $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ *is the equation of a line through the points* (x_1, y_1) *and* (x_2, y_2).

Example 12 *Use the previous example to find the equation of the line passing through the points* $(2, 3)$ *and* $(-1, 4)$.