

1 Section 10.4 Determinants

Associated with each square matrix A there is a number called the determinant of A . We will denote the determinant of the matrix A by $\det(A)$ or by $|A|$.

Determinant of a 2×2 Matrix

The determinant of the matrix $A = [a_{ij}]$ of order 2 is $|A| = a_{11}a_{22} - a_{21}a_{12}$.

Example 1 True or False: 1) $|A| = |A|$ 2) $|A|$ is a number.

Example 2 Find the value of the determinant of the matrix $A = \begin{bmatrix} 9 & 1 \\ 8 & 2 \end{bmatrix}$.

The Minor of a Matrix

The minor M_{ij} of the element a_{ij} of a square matrix A of order $n \geq 3$ is the determinant of the matrix of order $n - 1$ obtained by deleting the i th row and the j th column of A .

Example 3 Given the matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & -2 & 9 \\ 5 & -4 & 8 \end{bmatrix}$, find M_{13} .

Cofactor of a Matrix

The cofactor C_{ij} of the element a_{ij} of a square matrix A is given by $C_{ij} = (-1)^{i+j} M_{ij}$ is the minor of a_{ij} .

Example 4 Given $A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 4 & 0 \\ 2 & 1 & 1 \end{bmatrix}$, find M_{12} , C_{22} and C_{23} .

Determinants by Expanding by Cofactors

Given the square matrix A of order 3 or greater, the value of the determinant of A is the sum of the products of the elements of any row or column and their cofactors. For the r th row of A , the value of the determinant of A is $|A| = a_{r1}C_{r1} + a_{r2}C_{r2} + \cdots + a_{rn}C_{rn}$.

For the c th column of A , the determinant of A is $|A| = a_{1c}C_{1c} + a_{2c}C_{2c} + \cdots + a_{nc}C_{nc}$.

Example 5 Evaluate the determinant of $A = \begin{bmatrix} -2 & 3 & 1 \\ -3 & -4 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ by expanding by cofactors.

Effects of Elementary Row Operations on the Value of a Determinant of a Matrix

If A is a matrix of order n , then the following elementary row operations produce the indicated changes in the determinant of A .

1. Interchanging any two rows of A changes the sign of $|A|$.

- Multiplying a row of A by a constant k multiplies the determinant of A by k .
- Adding a multiple of a row of A to another row does not change the value of the determinant of A .

Example 6 *True or False:*

$$1. \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

$$2. \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$3. \begin{vmatrix} 1 & 4 \\ 8 & 12 \end{vmatrix} = 4 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$4. \begin{vmatrix} 4 & 8 \\ 12 & 16 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$5. \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix} = 4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$6. \begin{vmatrix} 4 & 8 \\ 12 & 16 \end{vmatrix} = 16 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$7. \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix}$$

These elementary row operations are often used to rewrite a matrix in triangular form. A matrix is in **triangular form** if all elements below or above the main diagonal are zero.

Example 7 $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 0 & -2 & 4 & 2 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 6 & 4 & -2 & 0 \\ 8 & 3 & 4 & 2 \end{bmatrix}$

are in triangular form.

Determinant of a Matrix in Triangular Form

Let A be a square matrix of order n in triangular form. The determinant of A is the product of the elements on the main diagonal.

$$|A| = a_{11}a_{22} \cdots a_{nn}.$$

Example 8 Find $|A|$ and $|B|$ from the previous example.

Example 9 Evaluate the determinant by rewriting it in triangular form $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 4 & 3 & 0 & 5 \\ -2 & -1 & 3 & -4 \\ 4 & 2 & 0 & 5 \end{bmatrix}$.

Conditions for a Zero Determinant

If A is a square matrix, then $|A| = 0$ when any one of the following is true.

1. A row (column) consists entirely of zeros.
2. Two rows (columns) are identical.
3. One row (column) is a constant multiple of a second row(column).

Example 10 Tell which condition for being a zero determinant is met by each

of the following determinants 1) $\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 3 \end{vmatrix}$ 2) $\begin{vmatrix} 5 & 1 & 3 \\ 2 & 2 & 6 \\ 6 & 1 & 3 \end{vmatrix}$ 3) $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix}$

Product Property of Determinants

If A and B are square matrices of order n , then $|AB| = |A||B|$.

Existence of the Inverse of a Square Matrix

If A is a square matrix of order n , then A has a multiplicative inverse if and only if $|A| \neq 0$. Furthermore, $|A^{-1}| = \frac{1}{|A|}$.

Example 11 Show that the determinant $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is the equation of a line through the points (x_1, y_1) and (x_2, y_2) .

Example 12 Use the previous example to find the equation of the line passing through the points $(2, 3)$ and $(-1, 4)$.