

1 Section 10.3 The Inverse of a Matrix

Multiplicative Inverse of a Matrix

If A is a square matrix of order n , then the inverse of matrix A , denoted by A^{-1} , has the property that $A \cdot A^{-1} = A^{-1} \cdot A = I_n$ where I_n is the identity matrix of order n .

Note that not all square matrices has a multiplicative inverse.

A procedure for finding the inverse uses elementary row operations. To the matrix A we will merge the identity matrix I to the right of A and denote this new matrix by $[A : I]$. Now, we will use elementary row operations to produce $[I : A^{-1}]$.

Example 1 Find A^{-1} if $A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$.

Example 2 Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 6 \\ -1 & -1 & 2 \end{bmatrix}$.

A **singular matrix** is a matrix that does not have a multiplicative inverse. A matrix that has a multiplicative inverse is a **non singular matrix**.

If there are all zeros in a row of the original matrix (after we apply the elementary row operations), then the matrix does not have an inverse.

Example 3 Show that the matrix $\begin{bmatrix} 1 & -6 & 4 \\ 3 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}$ is a singular matrix.

Systems of equations can be solved by finding the inverse of the coefficient matrix in the following steps:

- 1) Write the linear system as a matrix equation in the form $AX = B$.
- 2)The solution is $X = A^{-1}B$.

Example 4 Find the solution set of the following system of equations by using the inverse of the coefficient matrix:

$$1) \begin{cases} 3x_1 + 4x_2 = -1 \\ 3x_1 + 5x_2 = 1 \end{cases} \quad 2) \begin{cases} x_1 + 7x_3 = 20 \\ 2x_1 + x_2 - x_3 = -3 \\ 7x_1 + 3x_2 + x_3 = 2 \end{cases}$$

Example 5 True or False: 1) If $AB = O$, then $A = O$ or $B = O$. ($A = \begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 15 \\ -2 & 10 \end{bmatrix}$)

2) If A has an inverse and $AB = O$, then $B = O$.

3)If $AB = AC$, then $B = C$. ($A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & 7 \\ 3 & 11 \end{bmatrix}$)

4)If A has an inverse and $AB = AC$, then $B = C$.

Example 6 Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad-bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 7 Find the inverse (if possible) of 1) $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$ 2) $\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$.

Example 8 Show that $(AB)^{-1} = B^{-1}A^{-1}$.