1 Section 10.2 The Algebra of Matrices

Besides being convenient to be used for solving systems of equations, matrices are useful tools to model problems in business and science. Matrices are effective for situations in which there are a number of items to be classified.

Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if $a_{ij} = b_{ij}$ for every i and j.

Example 1 *Given* $\begin{bmatrix} 2 & a \\ b & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & x \\ 4 & y \end{bmatrix}$. *Determine the value of* a, b, x, y.

Addition of Two Matrices

If A and B are matrices of order $m \times n$, then the sum of the matrices is the $m \times n$ matrix given by $A + B = [a_{ij} + b_{ij}]$.

Additive Inverse of a Matrix

Given the matrix $A = [a_{ij}]$, the additive inverse of A is $-A = [-a_{ij}]$.

Example 2 *Given the matrices* $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$. *Find* 1)$A + B$ 2)*Additive inverse of* A.

Subtraction of Matrices

Given two matrices A and B of order $m \times n$, then $A - B$ is the sum of A and the additive inverse of B. $A - B = A + (-B)$

Example 3 *Find* $A - B$ *if* $A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}$.

Zero Matrix

Of special importance is the zero matrix, which is the matrix that consists of all zeros. The zero matrix is the additive identity for matrices and it is denoted by O.

Example 4 *Find* $\begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Given matrices A, B, C and the zero matrix O, each of order $m \times n$, then the following properties hold.

Commutative $A + B = B + A$ Associative $A + (B + C) = (A + B) + C$

Additive Inverse $A + (-A) = O$ Additive Identity $A + O = O + A = A$

Product of a Real Number and a Matrix

Given the $m \times n$ matrix $A = [a_{ij}]$ and the real number c, then $cA = [ca_{ij}]$.

Finding the product of a real number and a matrix is called scalar multiplication.

Example 5 *Determine cA if $c = 3$ and* $A = \begin{bmatrix} -3 & 2 \\ -2 & -6 \end{bmatrix}$.

Given real numbers a, b, and c and matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ each of order $m \times n$, then
$(b+c)A = bA + cA \qquad c(A+B) = cA + cB \qquad a(bA) = (ab)A$

Example 6 *Given* $A + \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, *find $3A - 2B$.*

In general, if A is a row matrix of order $1 \times n$, $A = [a_1 \ a_2 \ \cdots \ a_n]$ and B is a column matrix of order $n \times 1$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}$ then the product of A and B, written AB, is $AB = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

Example 7 *If* $A = \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}$, *then find AB.*

Product of Two Matrices

Let $A = [a_{ij}]$ be a matrix of order $m \times n$, and let $B = [b_{ij}]$ be a matrix of order $n \times p$. Then the product AB is the matrix of order $m \times p$ given by $AB = [c_{ij}]$, where each element c_{ij} is $cij = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

For the product of two matrices to be possible, the number of columns of the first matrix must equal the number of rows of the second matrix. The product matrix has as many rows as the first matrix and as many columns as the second matrix.

$A \qquad\qquad B \qquad\qquad = \qquad\qquad C$

Example 8 *True or False: 1) A 1×3 matrix can be multiplied by a single number. 2)A 1×3 matrix can be added to a 3×1 matrix.*

3) A 1×3 matrix can be multiplied by a 3×1 matrix. 4) A 2×4 matrix can be multiplied by a 2×4 matrix.

5) A 2×4 matrix can be multiplied by a 4×2 matrix.

Example 9 *If a matrix A is 4×3 and matrix B is 3×4, give the order of* 1)AB 2)BA.

Example 10 *Find each product:* 1) $\begin{bmatrix} 2 & 3 & -2 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 4 \\ 2 & 4 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$

Generally, matrix multiplication is not commutative. That is, given two matrices A and B, $AB \neq BA$.

Associative Property

Given matrices A, B, and C of orders $m \times n$, $n \times p$, and $p \times q$ respectively, then $A(BC) = (AB)C$.

Distributive property

Given matrices A_1 and A_2 of order $m \times n$ and matrices B_1 and B_2 of order $n \times p$,

then $A_1(B_1+B_2) = A_1B_1+A_1B_2 \qquad (A_1+A_2)B_1 = A_1B_1+A_2B_1$.

A square matrix that has a 1 for each element on the main diagonal and zeros elsewhere is called an **Identity Matrix** and it is denoted by I.

The identity matrix of order n, denoted I_n, is the $n \times n$ matrix $I_n = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & \cdot & 0 & \cdot \\ \cdot & 0 & \cdot & 0 \\ 0 & \cdot & 0 & 1 \end{bmatrix}$.

Multiplicative Identity Property of Matrices

If A is a square matrix of order n, and I_n is the identity matrix of order n, then $AI_n = I_nA = A$.

Example 11 *Consider the system of equations* $\begin{cases} 2x+3y-z=5 \\ x-2y+2z=6 \\ 4x+y-3z=5 \end{cases}$. *This system can be expressed as a product of matrices* $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 4 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$.

Example 12 *Given the matrices* $A = \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix}$, *find the* 3×2 *matrix* X *that is a solution of the equation* $2A - 3X = 5B$.

Example 13 *Use the matrices* $A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ *to find* 1)A^2 2)B^3.

Example 14 *Find the system of equations that is equivalent to the given matrix equation* $\begin{bmatrix} 2 & 7 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \end{bmatrix}$.