1 Section 10.1 Gaussian Elimination Method

A **matrix** is a rectangular array of numbers. Each number in a matrix is called an element. A matrix of m rows and n columns is said to be of **order** $m \times n$. A **square matrix** of order n is a matrix with n rows and n columns. We will use the notation a_{ij} to refer to the element of a matrix in the ith row and the jth column.

Example 1 *The matrix* $\begin{bmatrix} 2 & 5 & -2 & 5 \\ -3 & 6 & 4 & 0 \\ 1 & 3 & 7 & 2 \end{bmatrix}$ *is a* 3×4 *matrix with* $a_{23} = 4$, $a_{31} = 1$, *and* $a_{13} = -2$.

The elements a_{11}, $a_{22}, \cdots$, a_{mm} are the main diagonal of a matrix. The elements 2, 6, 7 form the main diagonal of the above matrix. A matrix can be created from a system of linear equations.

Example 2 *The system* $\begin{cases} 2x - 3y + z = 2 \\ x \qquad - 3z = 4 \\ 4x - y = 4z = 3 \end{cases}$ *can be written using only the coefficients and constants as* $\begin{bmatrix} 2 & -3 & 1 & 2 \\ 1 & 0 & -3 & 4 \\ 4 & -1 & 4 & 3 \end{bmatrix}$. *This matrix is called the augmented matrix of the system of equations.*

The matrix formed by the coefficients of the system is the **coefficient matrix, i.e.,** $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -3 \\ 4 & 1 & -4 \end{bmatrix}$. The matrix formed by the constants is called the **constant matrix, i.e.,** $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$. The **augmented matrix** is formed by the coefficients and the constant matrices.

Also, we can write a system of equations from an augmented matrix.

Example 3 $\begin{bmatrix} 2 & -1 & 4 & 3 \\ 1 & 1 & 0 & 2 \\ 3 & -2 & -1 & 2 \end{bmatrix} \longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow \begin{cases} 2x - y + 4z = 3 \\ x + y = 2 \\ 3x - 2y - z = 2 \end{cases}$.

In certain cases, an augmented matrix represents a system of equations that can be solved by back substitution.

Example 4 $\begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow \begin{cases} x - 3y + 4z = 5 \\ y + 2z = -4 \\ z = -1 \end{cases}$.

The solution by the back substitution is $(3, -2, -1)$. The matrix above is in echelon form.

Echelon Form

A matrix is in Echelon form if all the following conditions are satisfied.

1. The first nonzero number in any row is a 1.
2. Rows are arranged so that the column containing the first nonzero number in any row is to the left of the column containing the first nonzero number of the next row.
3. All rows consisting entirely of zeros appear at the bottom of the matrix.

Example 5 *Matrices in echelon form:*

$$1)\begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 2)\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix} \qquad 3)\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

We can write an augmented matrix in echelon form by using elementary row operations.

Elementary Row Operations

Given the augmented matrix for a system of linear equations, each of the following elementary row operations produces a matrix of an equivalent system of equations.

1. Interchanging any two rows. ($R_i \longleftrightarrow R_j$)
2. Multiplying all the elements in a row by the same nonzero number. (kR_i)
3. Replacing a row by the sum of that row and a nonzero multiple of any other row. $(kR_i + R_j)$

The Gaussian Elimination method is an algorithm that uses elementary row operations to solve a system of linear equations.

Example 6 *Solve* $\begin{cases} 2x+5y=-1 \\ 3x-2y=8 \end{cases}$

Example 7 *Tell three ways to obtain* 1 *at* a_{11} *in the matrix* $\begin{bmatrix} 7 & 14 & 7 & 35 \\ 2 & 4 & -3 & 5 \\ 1 & 0 & 4 & 6 \end{bmatrix}$.

Example 8 *Solve by using the Gaussian elimination method:*

$$1)\begin{cases} 2w+3x+4y+z=13 \\ 3w+2x+5y+2z=19 \\ 3w+x-4y+6z=-27 \\ w+x+y+z=2 \end{cases} \qquad 2)\begin{cases} x+4y+2z=-3 \\ 4x+5y+4z=-4 \\ 10x+7y+8z=-6 \end{cases}$$

$$3)\begin{cases} x-3y+2z=6 \\ 4x-y+3z=10 \\ 7x+y+4z=-2 \end{cases} \qquad 4)\begin{cases} t+u-v-w=4 \\ t-u+v-w=0 \\ 2t+u+2v+w=15 \end{cases}$$

Example 9 *Use the system of equations* $\begin{cases} x+3y-a^2z=a^2 \\ 2x+3y+az=2 \\ 3x+4y+2z=3 \end{cases}$. *Find all values of* a *for which the system of equation has: 1) a unique solution 2) infinitely many solutions 3) no solution.*

Example 10 *Find an equation of the plane that passes through the points* $(1,2,6)$, $(-1,1,7)$, *and* $(4,2,0)$. *Use the equation* $z=ax+by+c$.