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1 Section 4.2 Exponential Functions and Their Graphs

The exponential function f with base b is defined by $f(x) = b^x$ where $b > 0$, $b \neq 1$, and x is any real number.

Graphs of Exponential Functions:

For positive real numbers b, $b \neq 1$, the exponential function defined by $f(x) = b^x$ has the following properties:

1. f has the set of real numbers as its domain.
2. f has the set of positive real numbers as its range.
3. f has a graph with a y-intercept of $(0, 1)$.
4. f is a one-to-one function.
5. f has a graph asymptotic to the x-axis. If $b > 1$, $f(x) \to 0$ as $x \to -\infty$. If $0 < b < 1$, $f(x) \to 0$ as $x \to \infty$.
6. f is an increasing function if $b > 1$. f is a decreasing function if $0 < b < 1$.

For all real numbers x, the function defined by $f(x) = e^x$ is called the **natural exponential function**.

2 Section 4.3 Logarithmic Functions and Their Graphs

If $x > 0$ and b is a positive constant $(b \neq 1)$, then $y = \log_b x$ if and only if $b^y = x$. The function defined by $f(x) = \log_b x$ is a **logarithmic function** with base b. This function is the inverse of the exponential function $g(x) = b^x$.

Let $g(x) = b^x$ and $f(x) = \log_b x$ $(x > 0,\ b > 0,\ b \neq 1)$. Then $g(f(x)) = b^{\log_b x} = x$ and $f(g(x)) = \log_b b^x = x$.

The exponential form of $y = \log_b x$ is $b^y = x$.

The logarithmic form of $b^y = x$ is $y = \log_b x$.

Basic Logarithmic Properties

1) $\log_b b = 1$ 2) $\log_b 1 = 0$ 3) $\log_b(b^p) = p$

Graphs of Logarithmic Functions

For positive real numbers b, $b \neq 1$, the logarithmic function defined by $f(x) = \log_b x$ has the following properties:

1. f has the set of positive real numbers as its domain.
2. f has the set of real numbers as its range.
3. f has a graph with a x-intercept of $(1, 0)$.
4. f is a one-to-one function.

5. f has a graph asymptotic to the y-axis. If $b > 1$, $f(x) \to -\infty$ as $x \to 0^+$. If $0 < b < 1$, $f(x) \to \infty$ as $x \to 0^-$.

6. f is an increasing function if $b > 1$. f is a decreasing function if $0 < b < 1$.

Domains of Logarithmic Functions

The function $f(x) = \log_b g(x)$ has as its domain the set of all x for which $g(x) > 0$ provided that $b > 0$, and $b \neq 1$.

Common and Natural Logarithms

The function defined by $f(x) = \log_{10} x$ is called the **common logarithmic function**. It is customarily written without the base as $f(x) = \log x$.

The function defined by $f(x) = \log_e x$ is called the **natural logarithmic function**. It is customarily written as $f(x) = \ln x$.

3 Section 4.4 Properties of Logarithms

In the following properties, b, M, N are positive real numbers $(b \neq 1)$, and p is any real number.

Product property $\log_b MN = \log_b M + \log_b N$

Quotient property $\log_b \frac{M}{N} = \log_b M - \log_b N$.

Power property $\log_b (M^p) = p \log_b M$.

One-to-one property $\log_b M = \log_b N$ implies $M = N$.

Logarithm-of-each-side property $M = N$ implies $\log_b M = \log_b N$.

Inverse property $b^{\log_b P} = p$ (for $p > 0$) .

Change-of-base Formula

If x, a, and b are positive real numbers with $a \neq 1$, and $b \neq 1$, then $\log_b x = \frac{\log_a x}{\log_a b}$.

4 Section 4.5 Exponential and Logarithmic Equations

If a variable appears in an exponent of a term of an equation, then the equation is called an **exponential equation**.

If $b^x = b^y$, then $x = y$, provided that $b > 0$ and $b \neq 1$.

Equations that involve logarithms are called **logarithmic equations**. The properties of logarithms, along with the definition of a logarithm, are often used to find the solutions of a logarithmic equation.

5 Section 5.1 Angles and Arcs

An **angle** is formed by rotating a given ray about its endpoint to some terminal position. The original ray is the **initial side** of the angle, and the second ray is the **terminal side** of the angle. The common endpoint is the **vertex** of the angle.

One **degree** is the measure of an angle formed by rotating a ray $\frac{1}{360}$ of a complete revolution. The symbol for degree is $^\circ$.

Angles are often classified according to their measure.

180° angles are **straight angles**.

90° angles are **right angles**.

Angles that have a measure greater than 0° but less than 90° are **acute angles**.

Angles that have a measure greater than 90° but less than 180° are **obtuse angles**.

An angle is in **standard position** if its vertex is at the origin and its initial side is on the positive x-axis.

Two positive angles are **complementary angles** if the sum of the measures of the angles is 90°.

Two positive angles are **supplementary angles** if the sum of the measures of the angles is 180°.

If the terminal side of an angle in standard position lies on a coordinate axis, then the angle is classified as a **quadrantal angle**.

Given $\angle\theta$ in standard position with measure x°, then the measures of the angles that are **coterminal** with $\angle\theta$ are given by $x^\circ + k \cdot 360^\circ$ where k is an integer.

One **radian** is the measure of the central angle subtended by an arc of length r on a circle of radius r.

Given an arc of length s on a circle of radius r, the measure of the central angle subtended by the arc is $\theta = \frac{s}{r}$ radians.

To convert from radians to degrees, multiply by $\left(\frac{180^\circ}{\pi \text{ radinas}}\right)$

To convert from degrees to radians, multiply by $\left(\frac{\pi \text{ radians}}{180^\circ}\right)$.

Let r be the length of the radius of a circle and θ the nonnegative radian measure of a central angle of the circle. Then the **length of the arc** s that subtends the central angle is $s = r\theta$.

Linear speed v is distance traveled per unit time while **angular speed** ω is the angle through which a point on a circle moves per unit time.

6 Section 5.2 Trigonometric Functions of Acute Angles

Let θ be an acute angle of a right triangle. The values of the **six trigonometric functions** of θ are

$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$ $\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ $\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$

$\cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$ $\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$ $\csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$

Trigonometric Functions of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30°; \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45°; \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60°; \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

In the applications, an angle measured above the line of sight is called an **angle of elevation** and angle measured below the angle of sight is called an **angle of depression**.

7 Section 5.3 Trigonometric Functions of Any Angle

Let $P(x, y)$ be any point, except the origin, on the terminal side of an angle θ in standard position. Let $r = d(O, P)$, the distance from the origin to P. The six trigonometric functions of θ are

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}, x \neq 0$ $\csc\theta = \frac{r}{y}, y \neq 0$ $\sec\theta = \frac{r}{x}, x \neq 0$ $\cot\theta = \frac{x}{y}, y \neq 0$ where $r = \sqrt{x^2 + y^2}$.

Values of Trigonometric Functions for Quadrantal Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0°$	0	1	0	undefined	1	undefined
$90°$	1	0	undefined	1	undefined	0
$180°$	0	-1	0	undefined	-1	undefined
$270°$	-1	0	undefined	-1	undefined	0

Signs of the Trigonometric Functions

sign of	I	II	III	IV
$\sin\theta$ and $\csc\theta$	positive	positive	negative	negative
$\cos\theta$ and $\sec\theta$	positive	negative	negative	positive
$\tan\theta$ and $\cot\theta$	positive	negative	positive	negative

Reference Angle

Given $\angle\theta$ in standard position, its reference angle θ' is the smallest positive angle formed by the terminal side of $\angle\theta$ and the x-axis.

Reference Angle Theorem

To evaluate $\sin\theta$, determine $\sin\theta'$. Then use either $\sin\theta'$ or its opposite as the answer, depending on which has the correct sign.

8 Section 5.4 Trigonometric Functions of Real Numbers

Let W be the **wrapping function**, t be a real number, and $W(t) = P(x, y)$. Then

$\sin t = y$ $\cos t = x$ $\tan t = \frac{y}{x}$, $x \neq 0$ $\csc t = \frac{1}{y}$, $y \neq 0$ $\sec t = \frac{1}{x}$, $x \neq 0$ $\cot t = \frac{x}{y}$, $y \neq 0$.

Domain and Range of the Trigonometric Functions

Function	Domain	Range
$y = \sin t$	$\{t \mid -\infty < t < \infty\}$	$\{y \mid -1 \leq y \leq 1\}$
$y = \cos t$	$\{t \mid -\infty < t < \infty\}$	$\{y \mid -1 \leq y \leq 1\}$
$y = \tan t$	$\{t \mid -\infty < t < \infty, t \neq (2n+1)\pi/2\}$	$\{y \mid -\infty < y < \infty\}$
$y = \csc t$	$\{t \mid -\infty < t < \infty, t \neq n\pi\}$	$\{y \mid y \geq 1, y \leq -1\}$
$y = \sec t$	$\{t \mid -\infty < t < \infty, t \neq (2n+1)\pi/2\}$	$\{y \mid y \geq 1, y \leq -1\}$
$y = \cot t$	$\{t \mid -\infty < t < \infty, t \neq n\pi\}$	$\{y \mid -\infty < y < \infty\}$

The **odd trigonometric functions** are $y = \sin t$, $y = \csc t$, $y = \tan t$, and $y = \cot t$. The **even trigonometric functions** are $y = \cos t$ and $y = \sec t$.

The **period** of $\cos t$, $\sin t$, $\sec t$, $\csc t$ is 2π.

The **period** of $\tan t$ and $\cot t$ is π.

The **reciprocal identities** are $\sin t = \frac{1}{\csc t}$ $\cos t = \frac{1}{\sec t}$ $\tan t = \frac{1}{\cot t}$

The **ratio identities** are $\tan t = \frac{\sin t}{\cos t}$ $\cot t = \frac{\cos t}{\sin t}$

The **Pythagorean identities** are $\cos^2 t + \sin^2 t = 1$ $1 + \tan^2 t = \sec^2 t$ $1 + \cot^2 t = \csc^2 t$

9 Section 5.5 Graphs of the Sine and Cosine Functions

One cycle of the graph of $y = a \sin bx$ for both a and b positive has the following.

1. The amplitude is a.
2. The period is $\frac{2\pi}{b}$.
3. For $0 \leq x \leq \frac{2\pi}{b}$, the zeros are 0, $\frac{\pi}{b}$, $\frac{2\pi}{b}$.
4. The maximum value is a when $x = \frac{\pi}{2b}$, and the minimum value is $-a$ when $x = \frac{3\pi}{2b}$.
5. If $a < 0$, the graph is reflected across the x-axis.

One cycle of the graph of $y = a \cos bx$ for both a and b positive has the following.

1. The amplitude is a.
2. The period is $\frac{2\pi}{b}$.
3. For $0 \leq x \leq \frac{2\pi}{b}$, the zeros are $\frac{\pi}{2b}$, $\frac{3\pi}{2b}$.
4. The maximum value is a when $x = 0$, and the minimum value is $-a$ when $x = \frac{\pi}{b}$.
5. If $a < 0$, the graph is reflected across the x-axis.

10 Section 5.6 Graphs of the Other Trigonometric Functions

One cycle of the graph of $y = a \tan bx$ for both a and b positive has the following.

1. No amplitude.
2. The period is $\frac{\pi}{b}$.
3. $x = 0$ is a zero.
4. The graph passes through $\left(-\frac{\pi}{4b}, -a\right)$ and $\left(\frac{\pi}{4b}, a\right)$.
5. If $a < 0$, the graph is reflected across the x-axis.

One cycle of the graph of $y = a \cot bx$ for both a and b positive has the following.

1. No amplitude.
2. The period is $\frac{\pi}{b}$.
3. $x = \frac{\pi}{2b}$ is a zero.
4. The graph passes through $\left(\frac{\pi}{4b}, a\right)$ and $\left(\frac{3\pi}{4b}, -a\right)$.
5. If $a < 0$, the graph is reflected across the x-axis.

One cycle of the graph of $y = a \csc bx$ for both a and b positive has the following.

1. The period is $\frac{2\pi}{b}$.
2. The vertical asymptotes are the zeros of $y = a \sin bx$.
3. The graph passes through $\left(\frac{\pi}{2b}, a\right)$ and $\left(\frac{3\pi}{2b}, -a\right)$.
4. If $a < 0$, the graph is reflected across the x-axis.

One cycle of the graph of $y = a \sec bx$ for both a and b positive has the following.

1. The period is $\frac{2\pi}{b}$.
2. The vertical asymptotes are the zeros of $y = a \cos bx$.
3. The graph passes through $(0, a)$, $\left(\frac{\pi}{b}, -a\right)$ and $\left(\frac{2\pi}{b}, a\right)$.
4. If $a < 0$, the graph is reflected across the x-axis.

11 Section 5.7 Graphing Techniques

The graph of $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$, with $b > 0$ have
Amplitude: $|a|$ Period: $\frac{2\pi}{b}$ Phase Shift: $-\frac{c}{b}$
One cycle of each graph is completed on the interval $-\frac{c}{b} \le x \le -\frac{c}{b} + \frac{2\pi}{b}$

12 Section 6.1 Verification of Trigonometric Identities

13 Section 6.2 Sum, Difference, and Cofunction Identities

Sum and Difference Identities

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(\alpha + \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Cofunction Identities

$\sin(90^\circ - \theta) = \cos\theta$ $\cos(90^\circ - \theta) = \sin\theta$ $\tan(90^\circ - \theta) = \cot\theta$
$\cot(90^\circ - \theta) = \tan\theta$ $\sec(90^\circ - \theta) = \csc\theta$ $\csc(90^\circ - \theta) = \sec\theta$

14 Section 6.3 Double and Half Angle Identities

Double-Angle Identities

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$ $\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

Half-Angle Identities

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$ $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$ $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{1 - \cos\alpha}{\sin\alpha}$

15 Section 6.4 Functions of the form $f(x) = a\sin x + b\cos x$

$a\sin x + b\cos x = k\sin(x + \alpha)$ where $k = \sqrt{a^2 + b^2}$ and α is the angle for which $\sin\alpha = \frac{b}{k}$ and $\cos\alpha = \frac{a}{k}$.

16 Section 6.5 Inverse Trigonometric Functions

$y = \sin^{-1} x$ if and only if $x = \sin y$ where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
$y = \cos^{-1} x$ if and only if $x = \cos y$ where $-1 \le x \le 1$ and $0 \le y \le \pi$.

See Table 6.2 on page 506.
Composition of Trigonometric Functions and Their Inverses
If $-1 \le x \le 1$, then $\sin(\sin^{-1} x) = x$, and $\cos(\cos^{-1} x) = x$.
If x is any real number, then $\tan(\tan^{-1} x) = x$.
If $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then $\sin^{-1}(\sin x) = x$.
If $-0 \le x \le \pi$, then $\cos^{-1}(\cos x) = x$.
If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\tan^{-1}(\tan x) = x$.

17 Section 6.6 Trigonometric Equations

1. Factoring
2. Squaring Each side of an equation
3. Using the Quadratic Formula

18 Section 7.3 Vectors

A **vector** is a directed line segment. The length of the line segment is the magnitude of the vector, and the direction of the vector is measured by an angle.

Equivalent vectors have the same magnitude and the same direction.
Fundamental Vector Operations
If $v = \langle a, b\rangle$ and $w = \langle c, d\rangle$ are two vectors and k is a real number, then

1. $||v|| = \sqrt{a^2 + b^2}$
2. $v + w = \langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$
3. $kv = k\langle a, b\rangle = \langle ka, kb\rangle$

A **unit vector** is a vector whose magnitude is 1.
Definition of Unit Vectors i and j
$i = \langle 1, 0\rangle \qquad j = \langle 0, 1\rangle$
Horizontal and Vertical Components of a Vector
Let $v = \langle a_1, a_2\rangle$, where $v \ne 0$, the zero vector. Then $a_1 = ||v|| \cos\theta$ and $a_2 = ||v|| \sin\theta$ where θ is the angle between the positive x-axis and v.

The Horizontal component of v is $||v|| \cos\theta$. The Vertical component of v is $||v|| \sin\theta$.

Definition of Dot Product
Given $v = \langle a, b\rangle$ and $w = \langle c, d\rangle$, the dot product of v and w is given by $v \cdot w = ac + bd$.

Magnitude of a Vector in terms of the Dot Product
If $v = \langle a, b\rangle$, then $||v|| = \sqrt{v \cdot v}$
Alternative Formula for the Dot Product
If v and w are two nonzero vectors and α is the smallest non-negative angle between v and w, then $v \cdot w = ||v||||w|| \cos\alpha$.

Angle between Two Vectors

If v and w are two nonzero vectors and α is the smallest non-negative angle between v and w, then $\cos\alpha = \frac{v \cdot w}{||v||||w||}$ and $\alpha = \cos^{-1}\left(\frac{v \cdot w}{||v||||w||}\right)$.

Scalar Projection

If v and w are two nonzero vectors and α is the smallest non-negative angle between v and w, then the scalar projection of v on w, $\text{proj}_w v$, is given by $\text{proj}_w v = ||v|| \cos\alpha$.

Perpendicular Vectors

Two nonzero vectors v and w are orthogonal if and only if $v \cdot w = 0$.

19 Section 8.1 Parabolas

A **parabola** is the set of points in the plane that are equidistant from a fixed line (the Directrix) and a fixed point (the Focus) not on the directrix.

Standard Forms of the Equations of a Parabola with Vertex at (h, k)

Vertical Axis of Symmetry

The standard form of the equation of the parabola with vertex $V(h, k)$ and a vertical axis of symmetry is $(x-h)^2 = 4p(y-k)$.

The focus is $(h, k+p)$,and the equation of the directrix is $y = k - p$.

Horizontal Axis of Symmetry

The standard form of the equation of the parabola with vertex $V(h, k)$ and a horizontal axis of symmetry is $(y-k)^2 = 4p(x-h)$.

The focus is $(h+p, k)$,and the equation of the directrix is $x = h - p$.

20 Section 8.2 Ellipses

An **ellipse** is the set of all points in the plane, the sum of whose distances from two fixed points (Foci) is a positive constant.

Standard Forms of the Equation of an Ellipse with Center at (h, k)

Major Axis Parallel to the x-axis

The standard form of the equation of an ellipse with the center at (h, k) and major axis parallel to the x-axis is given by $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad a > b$. The length of the major axis is $2a$. The length of the minor axis is $2b$. The coordinates of the vertices are $(h+a, k)$ and $(h-a, k)$, and the coordinates of the foci are $(h+c, k)$ and $(h-c, k)$, where $c^2 = a^2 + b^2$.

Major Axis Parallel to the y-axis

The standard form of the equation of an ellipse with the center at (h, k) and major axis parallel to the y-axis is given by $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \qquad a > b$. The length of the major axis is $2a$. The length of the minor axis is $2b$. The coordinates of the vertices are $(h, k+a)$ and $(h, k-a)$, and the coordinates of the foci are $(h, k+c)$ and $(h, k-c)$, where $c^2 = a^2 + b^2$.

Eccentricity of an Ellipse

The eccentricity e of an ellipse is the ratio of c to a, where c is the distance from the center to a focus and a is one-half the length of the major axis. That is $e = \frac{c}{a}$. The eccentricity of an ellipse is a measure of its "roundness."

21 Section 8.3 Hyperbolas

A **hyperbola** is the set of all points in the plane, the difference between whose distances from two fixed points (Foci) is a positive constant.

Standard Forms of the Equation of a Hyperbola with Center at (h, k)

Transverse Axis Parallel to the x-axis

The standard form of the equation of a hyperbola with center at (h, k) and transverse axis parallel to the x-axis is given by $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. The coordinates of the vertices are $V_1\left(h + a, k\right)$ and $V_2\left(h - a, k\right)$. The coordinates of the foci are $F_1\left(h + c, k\right)$ and $F_2\left(h - c, k\right)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are $y - k = \pm\frac{b}{a}\left(x - h\right)$.

Transverse Axis Parallel to the y-axis

The standard form of the equation of a hyperbola with center at (h, k) and transverse axis parallel to the y-axis is given by $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. The coordinates of the vertices are $V_1\left(h, k + a\right)$ and $V_2\left(h, k - a\right)$. The coordinates of the foci are $F_1\left(h, k + c\right)$ and $F_2\left(h, k - c\right)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are $y - k = \pm\frac{a}{b}\left(x - h\right)$.

Eccentricity of a Hyperbola

The eccentricity e of a hyperbola is the ratio of c to a, where c is the distance from the center to a focus and a is the length of the semi transverse axis. That is $e = \frac{c}{a}$. The eccentricity of a hyperbola is a measure of its "wideness."

22 Section 9.1 Systems of Linear Equations in Two Variables

A **solution** of a system of equations in two variables is an ordered pair that is a solution of both equations.

The graphs of two linear equations in two variables can intersect at a single point, be the same line, or be parallel. When the graphs intersect at a single point or are the same line, the system is called a **consistent** system of equations. The system is called an **independent** system of equations when the lines intersect at exactly one point. The system is called a **dependent** system of equations when the equations represent the same line. In this case, the system has infinite number of solutions. When the graphs of the two equations are parallel lines, the system is called **inconsistent** and has no solution.

To solve a system of linear equations, we can use the **substitution method** or the **elimination method**. Two systems of equations are **equivalent** if each system has exactly the same solution.

Operations that produce Equivalent System of Equations

1. Interchange any two equations.
2. Replace an equation with a nonzero multiple of that equation.
3. Replace an equation with the sum of that equation and a nonzero constant multiple of another equation in the system.

23 Section 9.3 Nonlinear Systems of Equations

A **nonlinear system of equations** is one in which one or more equations of he system are not linear equations. To solve a nonlinear system of equations, use the substitution method or the elimination method.

24 Section 10.1 Gaussian Elimination Method

A **matrix** is a rectangular array of numbers. Each number in a matrix is called an element. A matrix of m rows and n columns is said to be of **order** $m \times n$. A **square matrix** of order n is a matrix with n rows and n columns.

A matrix can be created from a system of linear equations. The matrix formed by the coefficients of the system is the **coefficient matrix**. The matrix formed by the constants is called the **constant matrix**. The **augmented matrix** is formed by the coefficients and the constant matrices.

Echelon Form

A matrix is in Echelon form if all the following conditions are satisfied.

1. The first nonzero number in any row is a 1.
2. Rows are arranged so that the column containing the first nonzero number in any row is to the left of the column containing the first nonzero number of the next row.
3. All rows consisting entirely of zeros appear at the bottom of the matrix.

Elementary Row Operations

Given the augmented matrix for a system of linear equations, each of the following elementary row operations produces a matrix of an equivalent system of equations.

1. Interchanging any two rows.
2. Multiplying all the elements in a row by the same nonzero number.
3. Replacing a row by the sum of that row and a nonzero multiple of any other row.

The Gaussian Elimination method is an algorithm that uses elementary row operations to solve a system of linear equations.

25 Section 10.2 The Algebra of Matrices

Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if $a_{ij} = b_{ij}$ for every i and j.

Addition of Two Matrices

If A and B are matrices of order $m \times n$, then the sum of the matrices is the $m \times n$ matrix given by $A + B = [a_{ij} + b_{ij}]$.

Additive Inverse of a Matrix

Given the matrix $A = [a_{ij}]$, the additive inverse of A is $-A = [-a_{ij}]$.

Subtraction of Matrices

Given two matrices A and B of order $m \times n$, then $A - B$ is the sum of A and the additive inverse of B. $A - B = A + (-B)$

Zero Matrix

The $m \times n$ zero matrix, denoted by O, is the matrix whose elements are all zeros.

Product of a Real Number and a Matrix

Given the $m \times n$ matrix $A = [a_{ij}]$ and the real number c, then $cA = [ca_{ij}]$.

Product of Two Matrices

Let $A = [a_{ij}]$ be a matrix of order $m \times n$, and let $B = [b_{ij}]$ be a matrix of order $n \times p$. Then the product AB is the matrix of order $m \times p$ given by $AB = [c_{ij}]$, where each element c_{ij} is $cij = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

A square matrix that has a 1 for each element on the main diagonal and zeros elsewhere is called an **Identity Matrix** and it is denoted by I.

Multiplicative Identity Property of Matrices

If A is a square matrix of order n, and I_n is the identity matrix of order n, then $AI_n = I_nA = A$.

26 Section 10.3 The Inverse of a Matrix

Multiplicative Inverse of a Matrix

If A is a square matrix of order n, hen the inverse of matrix A, denoted by A^{-1}, has the property that $A \cdot A^{-1} = A^{-1} \cdot A = I_n$ where I_n is the identity matrix of order n.

A **singular matrix** is a matrix that does not have a multiplicative inverse. A matrix that has a multiplicative inverse is a **non singular matrix**.

Systems of equations can be solved by finding the inverse of the coefficient matrix. A linear system can be written in the form $AX = B$. The solution $X = A^{-1}B$.

27 Section 10.4 Determinants

Determinant of a 2×2 Matrix

The determinant of the matrix $A = [a_{ij}]$ of order 2 is $|A| = a_{11}a_{22} - a_{21}a_{12}$.

The Minor of a Matrix

The minor M_{ij} of the element a_{ij} of a square matrix A of order $n \geq 3$ is the determinant of the matrix of order $n-1$ obtained by deleting the ith row and the jth column of A.

Cofactor of a Matrix

The cofactor C_{ij} of the element a_{ij} of a square matrix A is given by $C_{ij} = (-1)^{i+j} M_{ij}$ is the minor of a_{ij}.

Determinants by Expanding by Cofactors

Given the square matrix A of order 3 or greater, the value of the determinant of A is the sum of the products of the elements of any row or column and their cofactors. For the rth row of A, the value of the determinant of A is $|A| = a_{r1}C_{r1} + a_{r2}C_{r2} + \cdots + a_{rn}C_{rn}$.

For the cth column of A, the determinant of A is $|A| = a_{1c}C_{1c} + a_{2c}C_{2c} + \cdots + a_{nc}C_{nc}$.

Effects of Elementary Row Operations on the Value of a Determinant of a Matrix

If A is a matrix of order n, then the following elementary row operations produce the indicated changes in the determinant of A.

1. Interchanging any two rows of A changes the sign of $|A|$.
2. Multiplying a row of A by a constant k multiplies the determinant of A by k.
3. Adding a multiple of a row of A to another row does not change the value of the determinant of A.

A matrix is in **triangular form** if all elements below or above the main diagonal are zero.

Determinant of a Matrix in Triangular Form

Let A be a square matrix of order n in triangular form. The determinant of A is the product of the elements on the main diagonal.

$|A| = a_{11}a_{22} \cdots a_{nn}$.

Conditions for a Zero Determinant

If A is a square matrix, then $|A| = 0$ when any one of the following is true.

1. A row (column) consists entirely of zeros.
2. Two rows (columns) are identical.
3. One row (column) is a constant multiple of a second row(column).

Product Property of Determinants

If A and B are square matrices of order n, then $|AB| = |A||B|$.

Existence of the Inverse of a Square Matrix

If A is a square matrix of order n, then A has a multiplicative inverse if and only if $|A| \neq 0$. Furthermore, $|A^{-1}| = \frac{1}{|A|}$.

This was done by Dr. Husain Alattas.