Inverse Problems are one of the most challenging problems in science.

Inverse Scattering Problems are used to determine inside character of a body. These arise in Radars, sonars, geological exploration, medical imaging, non destructive testing.

A signal is introduced that travels inside the body as waves. These waves interact with the discontinuity / inhomogeneity and gets *scattered*. From knowledge of this scattered field information about the inhomgeneity (reflector) is sought.

The signal could be acoustic, elastic or electromagnetic.

More often a direct problem is also needed to arrive at an *inversion* process. In direct problem material properties, boundary conditions and source is known and we determine the resulting wave field.

Direct Problems: (Scattering) Main methods of solution in this case are (Jones; Rawlins; Meister; Wickam)

- Greens Function
- Spectral Methods
- Wiener-Hopf Method
- Perturbation Method

Inverse Problems: (Bleistein; Colton; Kress; Cohen)

Problem is often reduced to a Fredholm integral equation of first kind

- Analytical Methods
- Regularization Methods
- Numerical Methods

Inverse Problem - An Introduction

Let us consider a wave propogating in an infinite medium of known wave speed $c\ x\ , x \quad 0$ and a small change in wave speed occurs for $x \quad 0$. Our objective is to determine the small change in the speed. The observable quantity $u\ x,$ - called field is assumed to satisfy a Helmotz field equation

$$\frac{d^2u}{dx^2} \qquad \frac{{}^2}{{}^2}u \qquad\qquad x$$

$$\frac{du}{dx} \qquad \frac{i}{\quad x}u \qquad 0 \text{ as } |x|$$

being wave velocity.

Let

$$\frac{1}{{}^2\ x} \qquad \frac{1}{c^2\ x}\ 1 \qquad x$$

$c\ x$ is background speed, x is small variation.

The governing eqaution becomes

$$£u \qquad\qquad x \qquad \frac{1}{c^2\ x} \qquad x$$

Let us write

$$u \ x \qquad u_0 \ x \qquad v \ x$$

$v \ x$ being scattered field $x \quad 0$ and $u_0$the incident field

u_0 satisfies

$$\frac{d^2 u}{dx^2} \quad \frac{2}{2} u \qquad x$$

$$\frac{du}{dx} \quad \frac{i}{x} u \quad 0 \text{ as } |x|$$

This leads to

$$£u \quad \frac{2}{c^2 \ x} \quad x \ u_0 \ x \quad v \ x$$

The solution to this equation can be written in terms of Green's function satisfying the homogenenous equation, so

$$v \ x \qquad 2 \quad \frac{x}{c^2 \ x} \ u_0 \ x \quad v \ x \ g \ x, \ , \quad dx$$

This is nonlinear and illposed integral eqaution in x .

Borns Approximation:

$$v\ x \qquad {}^{2} \quad \frac{x}{c^{2}\ x}\ u_{0}\ x \quad g\ x, \ \ , \qquad dx$$

If the medium in which the problem is set

has boundaries then the field is determined subject to the boundary conditions. The problems in seismology, ocean acoustics would require boundary conditions at the free surface. In most cases whole of the boundary is assumed to satisfy either Dirichlet, Neumann or Robin boundary conditions. The material of the medium is also usually assumed to be at rest. However for ocean acoustic direct or inverse problems or problems set in atmosphere the motion of tthe medium may also be of interest. For this purpose we shall assume the medium to be moving with velocty and will assume a two part boundary at ground or sea bed each having a different impedane constant.

Model:

We consider an acoustic medium occupying half space with z-axiz to be directed into the medium. The velocity of the medium is taken to be U in the direction parallel to x-axis.

1) The two part boundary is made up of e.g. land, forest, sea.

2) Each part of ground has a different constant impedance give by

$$Z \; x \qquad \begin{cases} Z_1, \; x \quad 0 \\ Z_2, \; x \quad 0 \end{cases}$$

3) The background sound speed c_0 is assumed to be constant.

4) THe object or inhomogeneity in the medium causes a small change in speed given by

$$\frac{1}{c^2 \; z} \qquad \frac{1}{c_0^2} \; 1 \qquad z$$

or $$\frac{1}{c \; z} \qquad \frac{1}{c_0}\left[1 \qquad \frac{z}{2}\right]$$

Boundary value Problem:The perturbation velocity of the irrotational sound wave can be written in terms of the velocity potential $u\ x,z$ as grad $u\ x,z$. The resulting pressure of the sound field is given by

$$p \qquad {}_0\left(\frac{}{t} \quad U\frac{}{x}\right)u,$$

where ${}_0$ is the density of the undisturbed stream. We shall restrict our consideration to the time harmonic variation $\exp \quad i \quad t$ and consider the configuration shown in Fig. 1. Then the problem reduces to one of solving the wave equation for moving fluid 8

$$\left\{ 1 \quad M^2 \ \frac{^2}{x^2} \quad \frac{^2}{z^2} \quad \frac{2i \ \ M}{c \ \ z} \frac{}{x} \quad \frac{^2}{c^2 \ \ z} \right\} u$$

$$x \quad a \quad z \qquad ,$$

subject to the boundary condition

$$\left\{ \frac{}{z} \quad \frac{0}{Z\,x} U \frac{}{x} \quad i \quad \frac{0}{Z\,x} \right\} u\;x,0$$

$0.$

Furthermore, the radiation condition for z , is also assumed. The radiation condition ensures that when z the total field behaves like a wave outgoing from the plane $z \quad 0$.

$$\left\{ 1 \quad M^2 \frac{2}{x^2} \quad \frac{2}{z^2} \quad \frac{2i\ \ M}{c_0} \frac{}{x} \quad \frac{2}{c_0^2} \right\} u_0\ x$$

$$x \quad a \quad z \qquad ,$$

with the above mentioned radiation condition for z and the boundary condition

$$\left\{ \frac{}{z} \quad \frac{0}{Z\ x} U \frac{}{x} \quad i \ \frac{0}{Z\ x} \right\} u_0\ x, 0$$

$$0.$$

The field $v\ x, z$ satisfies

$$\left\{ 1 \quad M^2 \frac{2}{x^2} \quad \frac{2}{z^2} \quad \frac{2i\ \ M}{c_0} \frac{}{x} \quad \frac{2}{c_0^2} \right\} v\ x,$$

$$\left\{ \frac{2 \quad z}{c_0^2} \quad \frac{i\ \ M \quad z}{c_0} \frac{}{x} \right\} u_0 \quad v \quad x, z \ ,$$

and

$$\left\{ \frac{}{z} \quad \frac{0}{Z\ x} U \frac{}{x} \quad i \ \frac{0}{Z\ x} \right\} v\ x, 0 \qquad 0,$$

Direct and Inverse Scattering Integral Equations

Let $v \quad ,z$ denote the Fourier transform of $v\ x,z$ with respect to x

$$v \quad ,z \qquad\qquad v\ x,z\ \ \exp \quad i\ \ x\ dx, \qquad\qquad ,$$

and the inverse transform yields

$$v\ x,z \qquad \frac{1}{2} \qquad v \quad ,z\ \ \exp\ i\ \ x\ \ d\ \ .$$

Where is a straight line in the regularity strip of $v \quad ,z$ in the complex plane. If $v \quad ,z$ has singularities on the real axes (as we shall see later that it has), then must pass above those singularities that lie on the real axis . Hence the narrow strip consists of the indented line .

Now let us take Fourier transform of the BVP with respect to x together with the convolution property of the Fourier transform we get

$$\frac{d^2 v}{dz^2} \quad {}_0^2 \quad v \quad \left[\frac{2}{c_0^2} \quad \frac{M}{c_0} \right] \quad z \quad u,$$

and

$$v \quad ,0 \quad \frac{i}{2 \quad {}_0}$$

$$\frac{}{z} \quad Z \quad v \quad ,z \; d \;\; {}_{z \; 0} \quad 0.$$

Where ${}_0$ denotes the following function

$$_0 \quad \sqrt{1 \quad M^2 \quad {}^2 \quad \frac{2 \; M}{c_0} \quad \frac{2}{c_0^2}},$$

$$_0 \; 0 \quad i \frac{}{c_0}.$$

If we set ${}_0 \quad 0$, then we find two roots namely $\frac{}{c_0 \; 1 \;\; M}$ and $\frac{}{c_0 \; 1 \;\; M}$. Since the flow is subsonic ($|M| \quad 1$), the second root always lie on negative real axes.

To solve these, we make use of Green's function $G \;\; z, \;\; , \;\;$, satisfying the following relations

$$\frac{d^2 G}{dz^2} \quad {}_0^2 \quad G \quad z \quad ,$$

$G \quad z, \quad ,$ outgoing wave as z ,

$$G \quad 0, \quad ,$$

$$\left\{ \frac{\quad}{z} \quad Z \quad G \quad z, \quad , \quad d \right\}_{z \; 0} \quad 0.$$

We follow the sam eprocedure as followed by Idemen and Akduman to arrive at the following operator equation

$$I \quad A \; D \quad f \quad , \quad ,$$

where I is a unit operator while the linear operator A and the known function f are defined by

$$AD \quad \frac{i \;_0 \quad \exp \;_0}{2 \quad _0 \quad U}$$

$$Z \quad D \quad \exp \quad _0 \quad d \; ,$$

and

$$f \quad , \qquad 1 \quad \frac{i \;_0 \qquad \exp \quad _0}{2 \quad _0 \qquad U}$$

$$Z \qquad \exp \quad _0 \qquad d \; .$$

The functional equation is the direct scattering equation and its solution procedure will be considered in the next section. Now, by assuming D and hence G is known, we can write solution $v \quad ,z$ as follows:

$$v \quad ,z \qquad \Big[\frac{\;\;^2}{c_0^2} \quad -$$

$$\int_0^h G \quad z, \quad , \qquad u \quad , \quad d \;, \qquad z \quad 0,$$

or it can be written in the operator form

$$I \quad B \quad v \quad B \; u_0 \; .$$

Where the operator B is defined by

$$B\,u \quad \left[\frac{2}{c_0^2} \qquad \frac{M}{c_0} \right]$$

$$\int_0^h G \;\; z, \;\; , \qquad\qquad u \qquad , \quad d \;\; .$$

From above equation we get

$$v \quad ,z \qquad I \quad B \;\; ^1 B\,u_0 \;\; .$$

This is the *inverse scattering equation* which connects the field with z appearing in B. This is the integral equation on which we base the inverse scattering theory. The field u_0 can be calculated in the same way as the field v as

$$u_0 \quad ,z \qquad G \;\; z, \;\; , \quad \exp \quad i \;\; a$$

$$\frac{\exp \quad i \;\; a}{2 \;\; _0}$$

$$\exp \quad _0 \quad |z \qquad | \quad D \quad \exp \quad _0$$

$$z, \qquad 0.$$

5. Solution of the direct scattering equation

The direct equation can be solved exactly by function-theoretic methods. The function $Z\ x$ has Fourier transform in the sense of distribution which can be obtained by defining

$$Z\ x \qquad Z_1 \qquad Z_2 \quad Z_1 \; H\ x$$

$$\qquad\qquad Z_1 \qquad Z_2 \quad Z_1 \quad \lim_{0} \quad \exp \qquad x\ H\ x \quad .$$

Here, $H\ x$ stands for Heaviside unit-step function defined by

$$H\ x \qquad \begin{cases} 1, & x \quad 0 \\ 0, & x \quad 0. \end{cases}$$

The Fourier transform of expression is given by

$$Z \qquad\qquad 2\ Z_1 \qquad\qquad i \frac{Z_2 \quad Z_1}{i0} .$$

Furthermore, the last term can also be written as

$$\frac{1}{\qquad i0} \qquad \frac{1}{\ } \quad i \qquad\qquad ,$$

This yields

$$Z \qquad Z_1 \quad Z_2 \qquad i\frac{Z_1 \quad Z_2}{\quad}.$$

Now WH equation with the help of above takes the following form

$$AD \qquad \frac{i\ _0 \qquad Z_1 \quad Z_2}{2\ _0 \qquad U} D \qquad \frac{_0 \qquad Z_1 \quad Z_2}{2 \quad _0 \qquad U}$$

$$\left(CP \qquad \frac{D \quad \exp \quad _0}{\quad} d \quad \right),$$

where CP stands for Cauchy principal value. Consider the functions f and f . The first is regular in the upper half plane and goes to zero as , while the second has same properties in the lower half plane.

We can write

$$\frac{1}{i} CP \qquad \frac{\exp \quad _0 \qquad D}{\quad} d \qquad \exp \quad _0 \qquad D$$

$$\exp \quad _0 \qquad D \qquad , \qquad ,$$

and

$\exp \quad {}_0 \quad D$

$\exp \quad {}_0 \quad D \quad \exp$

$\exp \quad {}_0 \quad \exp \quad {}_0 \quad \exp \quad {}_0$

Now the expression with the help of and takes the following form

$$AD \quad \frac{i_0 \quad \exp \quad {}_0}{{}_0 \quad U}$$

$$Z_1 \exp \quad {}_0 \quad D \quad Z_2 \exp \quad {}_0$$

Similarly

$$f \quad , \quad 1 \quad \frac{i_0 \quad \exp \quad {}_0}{{}_0 \quad U}$$

$$Z_1 \exp \quad {}_0 \quad Z_2 \exp \quad {}_0$$

Furthermore, we also need

$$D \quad \exp \quad {}_0$$

$$\exp \quad {}_0 \quad D \quad \exp \quad {}_0 \quad D$$

From the application point of view, in most cases the recorded data are high-frequency data, so we seek only a high-frequency solution to the direct and inverse problem. Mathematically, the high frequency approximation implies use of asymptotic methods to create high-frequency formulation of the forward and inverse problems. Owing to the high-frequency assumption, any result that can be expressed as a series of quantities multiplied by inverse powers of may be accurately approximated by the leading-order term(s) of the series. So by the use of binomial theorem we can easily write

$$\frac{1}{\quad U} \quad \frac{1}{\ }\left(1 \quad \frac{U}{\ }\right).$$

We get after some rearrangements the following Wiener-Hopf problem

$$\exp \quad 0$$

$$\frac{1 \quad \frac{iZ_2 \ 0}{}}{1 \quad \frac{iZ_1 \ 0}{}}$$

$$\frac{1 \quad \frac{iZ_1}{}}{1 \quad \frac{iZ_1}{}}$$

$$\frac{1 \quad \frac{iZ_2 \ _0 \qquad U}{_0 \ ^2}}{1 \quad \frac{iZ_1 \ _0 \qquad U}{_0 \ ^2}} \exp \quad 0 \qquad , \qquad .$$

The solution of this equation will be given in the next section.

5. Solution of the Wiener-Hopf problem

We use the Wiener-Hopf technique to solve equation. Following three cases must be treated separately:

1. The case $Z_1 \quad Z_2$.

2. The case $Z_1 \quad Z_2$ and $Z_1 Z_2 \quad 0$.

3. The case $Z_1 \quad 0$ and $Z_2 \quad 0$.

Now we analyze these three cases separately, and discuss the solution procedure in detail in the following three subsections.

5.1. The case $Z_1 \quad Z_2$

In equation , if we set $Z_1 \quad Z_2 \quad Z$, then we can easily get

$$D \qquad \frac{1 \quad \frac{iZ \;\; 0 \qquad U}{0 \;\; ^{2}}}{1 \quad \frac{iZ \;\; 0 \qquad U}{0 \;\; ^{2}}}.$$

This is the reflection coefficient in terms of impedance of the plane. Here we can also consider the case where impedance of the plane is zero, that is, the case $Z \quad 0$. In this case we can easily see that , $D \qquad 1$, which is the known result of the reflection coefficient of the perfectly absorbing plane.

5.2. The case $Z_1 \quad Z_2$ and $Z_1 Z_2 \quad 0$

In this case set

$$K \qquad \frac{1 \quad \frac{iZ_2 \;\; 0 \qquad U}{0 \;\; ^2}}{1 \quad \frac{iZ_1 \;\; 0 \qquad U}{0 \;\; ^2}},$$

Following the standard solution procedure $Noble$, the kernel has a factorization of the form $K \quad K \; K$. To perform such a factorization in a straightforward manner let us define the function

$$K_1 \quad ,Z \qquad 1 \quad \frac{i \;\; 0 \;\; ^2}{Z \;\; 0 \qquad U},$$

which permits us to write the kernel as

$$K \qquad \frac{Z_2}{Z_1} \frac{K_1 \quad ,Z_2}{K_1 \quad ,Z_1}.$$

The intermediate function $K_1 \quad ,Z$ defined above is continuous everywhere on the line . It has weak singularities at the points $\frac{}{c_0 \; 1 \;\; M}$, $\frac{}{c_0 \; 1 \;\; M}$ and $\frac{}{U}$, and also $K_1 \quad ,Z \qquad 1$ as on the line . This shows that the factors of K_1, that is, K_1 and K_1 can be obtained by the classical formulae

$$K_1 \qquad \exp\left\{\frac{1}{2\ i} \qquad \frac{\log K_1}{\qquad} d\ \right\}, \qquad \mathrm{Im} \qquad 0$$

$$K_1 \qquad K_1 \qquad , \qquad \mathrm{Im} \qquad 0.$$

Also note that the function K_1 is both regular and different from zero in the upper half plane. Furthermore, K_1 1 an . Similar properties hold for K_1 in the lower half plane. Now using the intermediate function and its factorization defined above, we can rearrange as follows:

$$\frac{K_1 \quad , Z_1}{K_1 \quad , Z_2}$$

$$\frac{Z_2}{Z_1}\frac{K_1 \quad , Z_2}{K_1 \quad , Z_1} \exp \quad {}_0 \qquad D \qquad f \ , \ ,$$

where

$$f \quad , \qquad \frac{K_1 \quad , Z_1}{K_1 \quad , Z_2}$$

$$\left\{ \frac{K_1 \quad , \quad Z_1}{K_1 \quad , Z_1} \exp \quad {}_0 \qquad \frac{Z_2}{Z_1} \frac{K_1}{K_1} \right.$$

Now we can apply Liouville's theorem which yields the unique solution of this expression as

$$\exp \quad {}_0 \qquad D \qquad \frac{K_1 \quad , Z_2}{K_1 \quad , Z_1} f \quad , \quad ,$$

$$\exp \quad {}_0 \qquad D \qquad \frac{Z_1}{Z_2} \frac{K_1 \quad , Z_1}{K_1 \quad , Z_2} f \quad , \quad .$$

Where $f \quad ,$ are such that $f \quad , \quad f \quad , \quad f \quad ,$ with $f \quad , \quad 0$ and $f \quad , \quad 0$ when in the upper and lower half plane respectively.The solution of the Wiener-Hopf problem can be found as

$$D \qquad \exp \quad {}_0$$

$$\left\{ \frac{K_1 \quad , Z_2}{K_1 \quad , Z_1} f \quad , \qquad \frac{Z_1}{Z_2} \frac{K_1 \quad , Z_1}{K_1 \quad , Z_2} f \quad , \right.$$

It is worthwhile to note that all the terms appearing on the right-hand side depend on the parameter , while the left-hand side is independent of this parameter. This fact can be used to check the accuracy of numerical computations.

5.3. The case $Z_1 \quad 0$ and $Z_2 \quad 0$

In this case the expression takes the form

$$\exp \quad {}_0 \qquad D$$

$$\frac{iZ_2 \;{}_0 \qquad U}{{}_0^{\;\;2}} K_1 \quad , Z_2 \quad \exp \quad {}_0 \qquad D$$

$$\exp \quad {}_0$$

$$\frac{iZ_2 \;{}_0 \qquad U}{{}_0^{\;\;2}} K_1 \quad , \quad Z_2 \quad \exp \quad {}_0 \qquad .$$

We can factorize the kernel as follows:

$0 \quad U \quad K_1 \quad , Z_2$

$$\left[\quad U \sqrt{\quad \overline{c_0 \; 1 \quad M}} \right.$$

$$\left[\sqrt{\quad \overline{c_0 \; 1 \quad M}} \; K_1 \right.$$

The solution can now be obtained by repeating all the steps of the previous case.

6. Solution of the inverse scattering equation

Now we consider the inverse equation and workout the details and conditions under which the inverse problem can be solved. First suppose that the Green's

function $G \; z, \; ,$ appearing in the equation is completely known. Also since the variation in wave speed z is small (less than one), therefore $| \; z \; |^2 \quad 1$. If we restrict to the strip $z \quad 0, h$, then the restriction of the operator B defined in is a bounded operator in $L^2 \; 0, h \; , C$. In fact, by Schwarz inequality we can write

$$\left\| B\ u \right\|_{L^2\ 0.h} \quad \left| \frac{2}{c_0^2} \quad \frac{M}{c_0} \right| \left\{ \int_0^h \int_0^h \right| G\ \ z, \ , \quad \Big|^2$$

or

$$B_{\ L^2\ 0.h} \quad \left| \frac{2}{c_0^2} \quad \frac{M}{c_0} \right| \left\{ \int_0^h \int_0^h \right| G\ \ z, \ , \quad \Big|^2 d$$

From this we conclude that by appropriately choosing the frequency ,
we can ensure the following inequality

$$B_{\ L^2\ 0.h} \quad 1.$$

Now we can use this inequality to expand the inverse operator in terms of Neumann series

$$v \quad ,z \qquad B \quad B^2 \quad B^3 \quad ... \quad u_0 \ , \qquad z \qquad 0,h \ .$$

If the frequency is such that $B \qquad 1,$
then square and higher powers of B can be neglected and so takes the form

$$v \quad ,z \qquad B\ u_0\ , \qquad z \qquad 0,h\ , \qquad .$$

This is well-known Born’s approximation.

This approximation stays valid if the following condition is met , namely

$$1 \quad \min \left[\frac{1}{\sqrt{\left| \frac{2}{c_0^2} \quad \frac{M}{c_0} \right|}} \left\{ \begin{matrix} h & h \\ 0 & 0 \end{matrix} \left| G \quad z, \quad , \quad \right|^2 d \ d \right. \right.$$

Now we set $z \quad b$ in the equation . The resulting equation involves as unknown function. Then the unknown function is the solution of the following integral equation

$$T \qquad {}^2_0 \qquad v \quad ,b \ , \qquad ,$$

where the linear operator T is defined by

$$T \quad \left(\frac{2}{c_0^2} \quad \frac{M}{c_0} \right) {}^2_0 \quad {}^h_0 \ G \quad b, \ , \qquad u_0$$

If the integral appearing on the right side is discretized by any one of the known quadrature techniques , then an equation involving some discrete values of the wave speed variation at some discrete points, say $_1, \ _2, \ _3, ..., \ _n$, is obtained

$$ {}^{2}_{0} \quad v \quad ,b \quad \overset{n}{\underset{j\ \ 1}{}} \quad T_j \quad {}_j, \qquad . $$

To determine the values of $_1, \ _2, \ _3, ..., \ _n$, we can consider *it* at certain points $_i \quad i \quad 1,2,...,n$, which yields a system of linear algebraic equations, namely

$$ {}^{2}_{0} \ {}_i \quad v \quad {}_i, b \quad \overset{n}{\underset{j\ \ 1}{}} \quad T_j \ {}_i \ {}_j, \qquad i \quad 1,2,3,...,n. $$

We can at least theoretically sove these for $_i$.