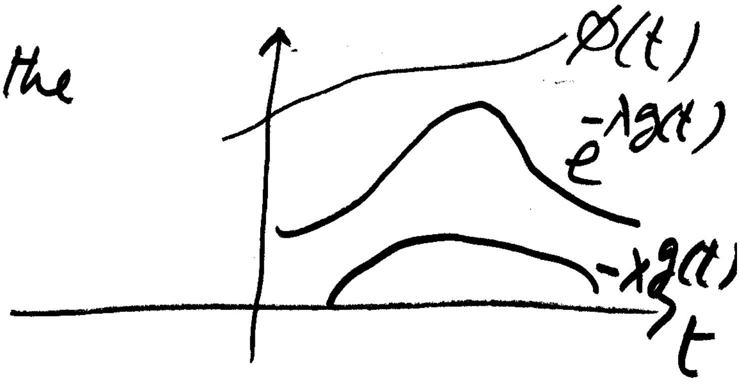
Method of Steepest Descent: *

Consider the integral $f(\lambda) = \int_a^b \phi(t)\, e^{-\lambda g(t)} dt$ —(1)

the integration being along the real axis. Let $g(t)$ be real and positive and $\phi(t)$ be slowly varying. We want asymptotic expression for $f(\lambda)$ for large λ.



Let us consider complex the complex counterpart

$$f(\lambda) = \int_\Gamma \phi(\zeta)\, e^{-\lambda g(\zeta)} d\zeta$$

$$\zeta = \xi + i\eta$$

$\phi(\zeta)$ and $g(\zeta)$ are analytic functions of ζ in a domain containing the path of integration. These functions are independent of λ.

Put $g(\zeta) = u(\xi,\eta) + i v(\xi,\eta)$

Integrand $= \phi(\xi + i\eta)\, e^{-\lambda u(\xi,\eta)}\, e^{-i\lambda v(\xi,\eta)}$

since λ is large, a small change in $v(\xi,\eta)$ will produce rapid oscillations. The integrand decays with $e^{-\lambda u(\xi,\eta)}$.

*(It is generalization of the Laplace method to the complex variables)

(S2

The main contribution to the integrand will come from the neighbourhood of $\zeta = \zeta_0$ where $u(\xi, \eta)$ is MINIMUM. We thus try to integrate along the path along which $u(\xi, \eta)$ decays most quickly (steepest descent). This is found that the path through $\zeta = \zeta_0$ defined by $v =$ constant is such a path.

Thus main idea is to deform the contour of integration into a contour on which $v =$ constant and which passes through $\zeta = \zeta_0$ where u is stationary.

The point ζ_0 is determined by

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \eta} = 0 \quad \text{(critical points)}$$

For minimum or max $\quad \dfrac{\partial^2 u}{\partial \xi^2} \dfrac{>0}{<0}$ and $\dfrac{\partial^2 u}{\partial \eta^2} \dfrac{>0}{<0}$

But since u is harmonic, $\dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 u}{\partial \eta^2} = 0$

Thus this point is neither max. nor min. but ~~a saddle point. By Cauchy - Riemann equation~~

a saddle point. By Cauchy-Riemann (S3

equation $\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}$, $\frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi}$

Thus at $\zeta = \zeta_0$, $\frac{dg}{d\zeta} = 0$. //

~~To determines~~ Now v = constant gives the path of steepest descent. To determine this use

$$\xi = s\cos\theta \quad , \eta = s\sin\theta$$

so that $\frac{\partial \xi}{\partial s} = \cos\theta$, $\frac{\partial \eta}{\partial s} = \sin\theta$

Now $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial s} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial s}$$

$$= u_\xi \cos\theta + u_\eta \sin\theta.$$

If this is to be maximum (steepest) for θ, we must have $\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial s}\right) = 0$

or $-u_\xi \sin\theta + u_\eta \cos\theta = 0$

Using Cauchy-Rieman Eqns, $-v_\eta \sin\theta - v_\xi \cos\theta = 0$

(S4

$$\frac{d^k g}{d\zeta^k}\Big|_{\zeta=\zeta_0} \neq 0 \text{ for min } k$$

$$\text{or } \left.-\frac{\partial v}{\partial s}\right|_{\zeta_1} = 0$$

Since ζ_1 is arbitrary, it follows that v is constant along the path of steepest descent.

By Taylor's series

$$g(\zeta) = g_0 + (\zeta-\zeta_0) g_1 + \frac{(\zeta-\zeta_0)^2}{2} g_2 + \cdots$$

$$\text{where } g_n = \left.\frac{d^n g}{d\zeta^n}\right|_{\zeta=\zeta_0}, \quad g_0 = g(\zeta_0)$$

since $\frac{dg}{d\zeta} = 0$, keeping terms up to $(\zeta-\zeta_0)^2$,

$$g(\zeta) = g_0 + \frac{(\zeta-\zeta_0)^2}{2} g_2$$

or $g(\zeta) - g_0 = \frac{1}{2}(\zeta-\zeta_0)^2 g_2$. Since $v =$ constant $g(\zeta) - g_0$ is real and so $\frac{1}{2}(\zeta-\zeta_0)^2 g_2$ is also real. Put $p^2 = \frac{1}{2}(\zeta-\zeta_0)^2 g_2$ then

$$f(\lambda) \sim \int_\Gamma \phi(\zeta)\, e^{-\lambda\left[g_0 + \frac{(\zeta-\zeta_0)^2}{2} g_2\right]} d\zeta$$

(S5

$$= e^{-\lambda g_0} \int_{\Gamma} \phi(\zeta)\, e^{-\lambda \frac{(\zeta-\zeta_0)^2}{2} g_2}\, d\zeta$$

$$= e^{-\lambda g_0} \int_{\Gamma} \phi(\zeta)\, e^{-\lambda p^2} \frac{d\zeta}{dp}\, dp$$

Since the exponential term decays very rapidly, we can write

$$f(\lambda) \simeq e^{-\lambda g_0} \int_{-\varepsilon}^{\varepsilon} \phi(\zeta)\, e^{-\lambda p^2} \frac{d\zeta}{dp}\, dp$$

$$= e^{-\lambda g_0} \phi(\zeta_0) \int_{-\varepsilon}^{\varepsilon} e^{-\lambda p^2} \frac{d\zeta}{dp}\, dp$$

$\arg\left[\frac{1}{2} r^2 e^{2i\theta_1} g_2\right]$

Put $\zeta - \zeta_0 = r e^{i\theta_1}$.

$g(\zeta) - g_0$ real $\Rightarrow$ $\arg[g(\zeta) - g_0] = 0$

$= \arg\left[\frac{1}{2}(\zeta-\zeta_0)^2 g_2\right] = \arg\left[g_2 e^{2i\theta_1}\right] = 0$

Also, $p^2 = \frac{1}{2} r^2 e^{2i\theta_1} g_2 = \frac{1}{2} r^2 |g_2|$ ✓

$\Rightarrow r = \dfrac{p}{\sqrt{\frac{1}{2}|g_2|}}$ Thus $\dfrac{d\zeta}{dp} = \dfrac{d\zeta}{dr}\dfrac{dr}{dp}$

$$= e^{i\theta_1} \frac{1}{\sqrt{\frac{1}{2}|g_2|}}$$

because $\arg\left[g_2 e^{2i\theta_1}\right] = 0$

~~should be real~~

$$\therefore f(\lambda) \sim e^{-\lambda g_0} \phi(\xi_0) \int_{-\varepsilon}^{\varepsilon} e^{-\lambda p^2} dp \cdot \frac{e^{i\theta_s}}{\sqrt{\frac{1}{2}|g_2|}}$$ (S6)

$$= e^{-\lambda g_0} \phi(\xi_0) \frac{e^{i\theta_s}}{\sqrt{\frac{1}{2}|g_2|}} \int_{-\varepsilon}^{\varepsilon} e^{-\lambda p^2} dp$$

We can extend the limit to $(-\infty, \infty)$

so $$f(\lambda) \sim e^{-\lambda g_0} \phi(\xi_0) \frac{e^{i\theta_s}}{\sqrt{\frac{1}{2}|g_2|}} \int_{-\infty}^{\infty} e^{-\lambda p^2} dp$$

$$= e^{-\lambda g_0} \phi(\xi_0) \frac{e^{i\theta_s}}{\sqrt{\frac{1}{2}|g_2|}} \sqrt{\frac{\pi}{\lambda}}$$

<u>How to apply</u>: 1 Find ξ_0 for which $\frac{dg}{d\xi} = 0$

2 Expand $g(\xi)$ in Taylor series

$$g(\xi) = g(\xi_0) + \underline{(\xi - \xi_0)} g_1 + \frac{1}{2}(\xi - \xi_0)^2 g_2 + \cdots$$

3 Use $\arg\left[\frac{1}{2} g_2 e^{2i\theta_s}\right] = 0$ to get θ_s