Singular Perturbation Methods

For ε a small real parameter, $|\varepsilon| << 1$, the boundary value problem $L(u, x, t, \varepsilon) = 0$

with boundary conditions

$$B(u, x_0, t, \varepsilon) = 0$$

could be solved using perturbation method if expansion

$$u(x,t,\varepsilon) = u_0(x,t) + \varepsilon\, u_1(x,t) + \cdots$$

was valid. Such a problem is a regular perturbation problem. In many practical situations $u(x,t;\varepsilon)$ is not analytic as $\varepsilon \to 0$ and so above expansion is not uniformly valid. Such a problem is known as a singular perturbation problem.

Example: Consider $\quad \varepsilon u'' + u' = 0, \quad |\varepsilon| << 1 \quad$ —(1)

$$0 \le x \le 1.$$

Boundary conditions to be specified

We can write exact solution as

$$u(x;\varepsilon) = a_1(\varepsilon)\, e^{-x/\varepsilon} + a_2(\varepsilon)$$

where a_1, a_2 are constants but may depend upon parameter ε.

Now consider boundary conditions

$$u(0;\varepsilon) = U_0, \quad u(1;\varepsilon) = U_1 \qquad (U_0 \neq U_1)$$

a_1 and a_2 can be determined

$$a_1 = \frac{U_0 - U_1}{1 - e^{-1/\varepsilon}}$$

$$a_2 = \frac{U_1 - U_0\, e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}$$

which can lead to solution (some simplification is required)

$$u(x;\varepsilon) = \frac{U_0\left[e^{(1-x)/\varepsilon} - 1\right]}{\left[e^{1/\varepsilon} - 1\right]} + \frac{U_1\left[1 - e^{-x/\varepsilon}\right]}{\left[1 - e^{-1/\varepsilon}\right]} \quad -(2)$$

We notice that $\lim_{\varepsilon \to 0} u(x;\varepsilon)$ does not exist and so a power series expansion around 0 is not valid.

Let us distinguish between cases

(i) $\varepsilon \to 0$ from right i.e $\varepsilon > 0$ and write it as $\varepsilon \downarrow 0$

(ii) $\varepsilon \to 0$ from left i.e $\varepsilon < 0$ and write it as $\varepsilon \uparrow 0$.

From above solution as $\varepsilon \downarrow 0$ ($\boxed{-x \sim 1-x}$

$$u(x;\varepsilon) \sim U_0 e^{-x/\varepsilon} + U_1, \quad 0 < x \leq 1$$
$$0 < \varepsilon << 1$$

$$\to U_1, \quad 0 < x \leq 1 \text{ as } \varepsilon \downarrow 0 \quad (3$$

For a negative ε, $\varepsilon \uparrow 0$

$$u(x;\varepsilon) \sim U_0 + U_1 e^{(1-x)/\varepsilon} \quad 0 < x \leq 1$$
$$0 < \varepsilon << 1$$

$$\to U_0, \quad 0 \leq x < 1 \text{ as } \varepsilon \uparrow 0 \quad (4)$$

(5

Now consider solution in the limiting case $\varepsilon \downarrow 0$ (ε +ve). We notice that this satisfies the boundary condition at $x=1$ (but can not satisfy it at 0). If we let $\varepsilon \downarrow 0$ in the differential equation, it gives

$$u' = 0 \implies u(x) = \text{Constant}$$

and this can satisfy $u(1) = U_1$ if this constant is U_1 and $u(0) = U_0$ if this constant is U_0.

Such a problem can be foreseen if ε appears with the highest derivative in the differential equation.

Again from $\varepsilon > 0$ case, near $x=0$, we get from (2)

$$u(x;\varepsilon) \sim U_0 e^{-x/\varepsilon} + U_1(1 - e^{-x/\varepsilon}) \text{ as } \varepsilon \downarrow 0 \quad \text{—(5)}$$

(Multiply & divide first term by $e^{-1/\varepsilon}$ and see)

which shows that

$$\lim_{x \downarrow 0} \lim_{\varepsilon \downarrow 0} u(x;\varepsilon) = U_1 \neq U_0 = \lim_{\varepsilon \downarrow 0} \lim_{x \downarrow 0} u(x;\varepsilon)$$

Thus the order of limit is not commutative.

From (5) we note that in the NHD of $x=0$, there is a "boundary layer" near $x=0$ that is $O(\varepsilon)$ in which solution changes very rapidly from the boundary value U_0 to U_1 (as $\lim_{\substack{\varepsilon \downarrow 0 \\ x \neq 0}} u(x;\varepsilon) = U_1$)

From (2)

$$u'(0;\varepsilon) = \frac{1}{\varepsilon}(U_1 - U_0)$$

so gradient of u at $x=0$ is $O(1/\varepsilon)$.

On the other hand when $\varepsilon \uparrow 0$ (negative ε), the singular region is at $x=1$. (We can follow the same steps as above at $x=1$)

If we naively assume

$$u(x;\varepsilon) \sim u_0(x) + \varepsilon u_1(x) + \cdots \quad \text{as } \varepsilon \to 0$$

and put it in $\varepsilon u'' + u' = 0$. Comparing coeff of ε give

$$u_0' = 0, \quad u_n' = -u_{n-1}'', \quad n \geq 1$$

$$\Rightarrow u_n' = 0, \quad n \geq 0$$

which means $u_n = a_n$ (const)

resulting in

$$u(x;\varepsilon) \sim a_0 + \varepsilon a_1 + \cdots$$

which cannot satisfy both boundary conditions

<u>What to do</u>: Suppose we choose the solution to satisfy boundary condition at $x=1$.

This gives $a_0 = U_1$, $a_n = 0$ $n \geq 1$.

i.e $u(x;\varepsilon) \sim U_1$.

Now introduce $\xi = x/\varepsilon^{\alpha}$

α to be determined.

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{1}{\varepsilon^{\alpha}}\frac{du}{d\xi}$$

This transform stretches out the immediate NHD of $x=0$ since for any positive x, ξ is large because ε is small. For $x>0$, $\xi \to \infty$ as $\varepsilon \downarrow 0$. imp

SPS

Write $u(x;\varepsilon) = \bar{u}(\xi;\varepsilon)$ and D.E becomes

$$\varepsilon^{1-2\alpha}\,\bar{u}_{\xi\xi} + \varepsilon^{-\alpha}\,\bar{u}_{\xi} = 0$$

or

$$\varepsilon^{1-\alpha}\,\bar{u}_{\xi\xi} + \bar{u}_{\xi} = 0$$

We wish to determine α in such a way that as $\varepsilon \downarrow 0$ with fixed ξ, the second order term remains in the first approximation. This can be achieved by $\alpha = 1$. In this simple case, the equation becomes free of ε for $\alpha = 1$. This equation governs the solution in the NHD of $x=0$

We solve it subject to boundary conditions.

$\bar{u}(0;\varepsilon) = U_0$ and joins to $u(x;\varepsilon) = U_1$ outside

Such a $\bar{u}(\xi,\varepsilon)$ solution is $\bar{u}(\xi;\varepsilon) = U_1 + (U_0 - U_1)\,e^{-\xi}$

in terms of original variables

$$u(x;\varepsilon) \sim U_1 + (U_0 - U_1)\,e^{-x/\varepsilon} \text{ as } \varepsilon \downarrow 0$$

This is the same as exact solution

Similar procedure can be followed by choosing start with a solution satisfying B.C at $x=0$.

Initial value Problem:

$$u(0;\varepsilon) = U_0, \quad u'(0;\varepsilon) = V_0$$

where U_0 and V_0 are independent of ε.

If we again try