

CONNECTION SYMBOLS IN DIFFERENTIAL AND RIEMANNIAN GEOMETRY

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Abstract: Whereas algebraic softwares have made many lengthy and tedious calculations possible through various packages, it is quite difficult to write one's own codes without having much of a programming skill. However, from ones experience with programming in Mathematica it can be realized that writing most of the codes in mathematica language does not demand lot of programming skill. With this point in mind, and the fact that algebraic computational techniques are vital for most of us to know, we have tried to write a Mathematica based code for beginners in algebraic computations who lack or have very little programming skill. This code, though written extremely naively, is an extremely powerful tool and can evaluate connection symbols that arise in Differential and Riemannian geometry in an arbitrary dimension greater than 2.

Keywords: Algebraic computations, Geometry

Introduction

An n-dimensional Riemannian manifold is a set M together with a choice (for each point $p \in M$) of an inner product \langle, \rangle_p in the tangent space $T_p(M)$ such that for a given parameterization $X_\alpha: U_\alpha \rightarrow M$ with $p \in X_\alpha(U)$ the functions

$$g_{ij}(u_1, \dots, u_n) = \langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \rangle \quad (1)$$

are differentiable at $x_\alpha^{-1}(p)$. The (u_1, \dots, u_n) are called the coordinates of the open set $U_\alpha \subset R^n$. In this notation of the inner product the g_{ij} represents a Riemannian metric [1]

$$g_{ij} = \langle X_i, X_j \rangle \quad (2)$$

where $X_i = \frac{\partial}{\partial u_i}$. In this new basis the usual derivative operator gets modified to define a new derivative, called covariant derivative. This derivative differs

from the usual derivative (defined in terms of the basis $\{e_i\}$) by the term

$$D_{x_i} X_j = C_{ij}^k X_k \quad (3)$$

In the above equation the C_{jk}^i is known as Christoffel symbols and the Einstein summation convention is employed on the repeated index "k". Requiring that the manifold is torsion free, the C_{jk}^i becomes a symmetric tensor in its lower indices. At this stage what is left is to evaluate the C_{jk}^i consistent with the covariant derivatives. For this purpose we differentiate equation (2) with respect to the coordinate u_k to get,

$$\frac{\partial}{\partial u_k} \langle X_i, X_j \rangle = \langle D_{X_k} X_i, X_j \rangle + \langle X_i, D_{X_k} X_j \rangle \quad (4)$$

Cyclically permuting indices in equation (4), we get two more equations:

$$\frac{\partial}{\partial u_i} \langle X_j, X_k \rangle = \langle D_{X_i} X_j, X_k \rangle + \langle X_j, D_{X_i} X_k \rangle \quad (5)$$

$$\frac{\partial}{\partial u_j} \langle X_k, X_i \rangle = \langle D_{X_j} X_k, X_i \rangle + \langle X_k, D_{X_j} X_i \rangle \quad (6)$$

Now adding equations (4) and (5) and subtracting equation (6) instantly gives

$$\begin{aligned} \frac{1}{2} \left[\frac{\partial}{\partial u_k} \langle X_i, X_j \rangle + \frac{\partial}{\partial u_i} \langle X_j, X_k \rangle - \right. \\ \left. \frac{\partial}{\partial u_j} \langle X_k, X_i \rangle \right] = \langle D_{X_k} X_i, X_j \rangle \end{aligned} \quad (7)$$

Writing equation (7) in terms of the metric and its derivative, it becomes

$$C_{ki}^m g_{mj} = \frac{1}{2} \left[\frac{\partial g_{ij}}{\partial u_k} + \frac{\partial g_{jk}}{\partial u_i} - \frac{\partial g_{ki}}{\partial u_j} \right] \quad (8)$$

Since the requirement on the metric is that it must be non-degenerate (its determinant being non-zero) over the entire manifold, its inverse exist such that $g_{ij} g^{jk} = \delta_i^k$. Thus, operating on equation (8) by g^{pj} and renaming the dummy index gives,

$$C_{ik}^p = \frac{1}{2} g^{pj} \left[\frac{\partial g_{ij}}{\partial u_k} + \frac{\partial g_{jk}}{\partial u_i} - \frac{\partial g_{ki}}{\partial u_j} \right] \quad (9)$$

The C_{ik}^p is called Christoffel symbol. This symbol appears in many calculations in Geometry where we use non-Cartesian coordinates. In n-dimensions it has a total of n^3 components. Thus, whereas it is easy to compute this symbol in 2 or 3 dimensions, it becomes highly tedious to evaluate components of the Christoffel symbols in higher dimensions but it is quite an easy task to deal with such situations if one can use algebraic computations for this purpose. However, it is not always possible to have the ready made routines available that can be used in situations like this. Thus, it is of great use if one can write small routines to algebraically compute such expressions. Nevertheless, these routines can be written only when one has a reasonable knowledge of algebraic programming at the back of ones mind [2,3].

In the next section we give our routine, using commonly used commands of Mathematica that computes Christoffel symbols in any given metric of dimension greater than 2. This is then followed by stepwise procedure to load and use this routine, and lastly by two illustrative examples to show how this routine works for the metrics representing surface of a sphere of some constant radius and in 4-dimensional of the Schwarzschild metric [4].

Routine for computing Christoffel Symbols

```
M[dim, coordinates, components]:=
Block[{d},
dimension:=dim;
metric coordinates:=coordinates;
metric=components;
metricup=Inverse[metric];
metric derivative[a,b,c]:=D[metric[[a,b]],
metric coordinates[[c]]];
Chris[a,b,c]:=Simplify[Sum[(1/2)*(metricup[[a
,d]])*(metric derivative[b,d,c]+
Metric derivative[c,d,b]-metric
derivative[b,c,d]),{d,1,dimension}]]];
```

Illustration of Computing Christoffel Symbols

Example 1

The metric for the surface of a unite sphere is given by [1]

$$ds^2 = dr^2 + r^2 \sin^2 \theta d\theta. \quad (10)$$

To compute the Christoffel symbols “ C_{ij}^k ” for the above metric we perform the following steps:

Step 1

Open a note book in Mathematica and type the code as given in “Routine for computing Christoffel symbols”.

Step 2

Load this routine in Mathematica note book by pressing shift and enter keys together.

Step 3

Insert the metric given in Eq. (10) at the end of the code in the following format:

M[2, {t, th}, {{1,0},{0,r^2}}] and load it again.

Step 4

Write Chris [i, j, k] for i, j, k= 1, 2 to compute all of the C^k_{ij} in fraction of a second (1 and 2 respectively represent ϑ and φ coordinates). A copy of input and output command for computations of all the Christoffel symbols C^k_{ij} for i, j, k=1,...,2 is listed below.

```
In[1]: Chris[1,1,1]
Out 0
In[2]: Chris[1,1,2]
Out 0
In[3]: Chris[1,2,2]
Out -r
In[4]: Chris[2,1,1]
Out 0
In[5]: Chris[2,2,1]
Out 1/r
In[6]: Chris[2,2,2]
Out 0
```

Example 2:

We now consider a 4- dimensional Schwarzschild metric of general relativity given by [4],

$$ds^2 = (1 - 2m/r)dt^2 - (1 - 2m/r)^{-1}dr^2 - r^2d\vartheta^2 - r^2\sin^2\theta d\phi^2 \quad (11)$$

To compute the Christoffel symbols " C^k_{ij} " for the above metric, we perform the following steps once again.

Step 1

Open the note book in Mathematica once again and type the code as given in "Routine for computing Christoffel symbols".

Step 2

Load this code in the same Mathematica note book by pressing shift and enter keys together.

Step 3

Insert the metric given in Eq. (11) at the end of the code in the following format.

M[4, {t, r, th, phi}, {{(1-2*m/r), 0, 0, 0}, {0, -(1-2*m/r)^-1, 0, 0}, {0, 0, -r^2, 0}, {0, 0, 0, -r*Sin[th]^2}}] and load it again.

Step 4

Write Chris[i, j, k] for i, j, k= 1, ...,4 to compute all of the C^k_{ij} (1, 2, 3, 4, respectively, represent t, r, ϑ and φ coordinates). A copy of input and output command for computations of all of non-zero Christoffel symbols C^k_{ij} for i, j, k=1,..., 4 is listed below.

```
In[1]: Chris[1,1,1]
Out  $\frac{m}{(r^2 - 2mr)}$ 
In[2]: Chris[2,1,1]
Out  $\frac{m(r - 2m)}{r^3} 0$ 
In[3]: Chris[2,2,2]
Out  $\frac{m}{(2mr - r^2)}$ 
In[4]: Chris[2,3,3]
Out  $2m - r$ 
In[5]: Chris[2,4,4]
Out  $(2m - r)\sin^2 \vartheta$ 
In[6]: Chris[3,2,3]
Out  $1/r$ 
In[7]: Chris[3,4,4]
Out  $-\sin(2\vartheta)/2$ 
In[8]: Chris[4,2,4]
Out  $1/r$ 
In[9]: Chris[4,3,4]
Out  $\cot \vartheta$ 
```

It is interesting to note that despite being extremely simple, the routine is quite powerful in its own right as it can evaluate Christoffel symbols in any coordinated system in any arbitrary dimension greater than 2.

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