Mellin Transforms.

Mellin transform are also useful in some applications including theory of primes in number theory.

Motivation. Consider complex Fourier transform

$$\mathcal{F}\{g(\xi)\} = g^*(\alpha) = \int_{-\infty}^{\infty} g(\xi)\, e^{-i\alpha\xi}\, d\xi$$

$$\mathcal{F}^{-1}\{g^*(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} g^*(\alpha)\, e^{i\alpha\xi}\, d\alpha$$

Put $e^{\xi} = x$ and $i\alpha = c - p$
(c constant)

$$g^*(ip - ic) = \int_0^{\infty} x^{p-c-1}\, g(\log x)\, d\xi$$

$$g(\log x) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} x^{c-p}\, g^*(ip - ic)\, dp$$

We use these integrals (with some adjustment) to define Mellin transform

$$\mathcal{M}\{f(x)\} = \tilde{f}(p) = \int_0^{\infty} x^{p-1} f(x)\, dx$$

$$\mathcal{M}^{-1}\{\tilde{f}(p)\} = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\, \tilde{f}(p)\, dp$$

where $f(x)$ is a real valued function defined on $(0,\infty)$ and p is a complex variable.

Examples: ① $f(x) = e^{-nx}$, $n > 0$

$$\mathcal{M}\{f(x)\} = \tilde{f}(p) = \int_0^{\infty} x^{p-1} e^{-nx}\, dx$$

(M₂

Put $nx = t$,

$$\mathcal{M}\{f(x)\} = \frac{1}{n^p}\int_0^\infty t^{p-1} e^{-t}\,dt = \frac{\Gamma(p)}{n^p}.$$

Example 2 $\quad f(x) = \frac{1}{1+x}$

$$\mathcal{M}\left\{\frac{1}{1+x}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1}\frac{dx}{x+1}$$

Put $x = \frac{t}{1-t}$ n $t = \frac{x}{1+x}$

$$dx = \frac{1-t+t}{(1-t)^2}dt = (1+x)\,dt$$

Thus $\tilde{f}(p) = \int_0^1 t^{p-1}(1-t)^{-p}\,dt$

$$= B(p, 1-p)$$

(can be shown to be $\pi \csc(\pi p)$)

Example 3 (Riemann Zeta function)

$f(x) = (e^{x}-1)^{-1}$ then

$$\mathcal{M}\left\{\frac{1}{e^{x}-1}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1}\frac{1}{e^{x}-1}\,dx$$

Now $\frac{1}{1-e^{-x}} = \sum_{n=0}^{\infty} e^{-nx} \Rightarrow \sum_{n=1}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}} - 1 = \frac{1}{e^{x}-1}$

So, $\tilde{f}(p) = \sum_{n=1}^{\infty}\int_0^\infty x^{p-1}e^{-nx}\,dx = \sum_{n=1}^{\infty}\frac{\Gamma(p)}{n^p}$

We define $\zeta(p) = \sum_{n=1}^{\infty}\frac{1}{n^p}$, $(\mathrm{Re}\,p > 1)$

$$\tilde{f}(p) = \Gamma(p)\,\zeta(p).$$

Properties:

① $\mathcal{M}\{f(ax)\} = a^{-p}\tilde{f}(p), \ a>0$

$\mathcal{M}\{f(ax)\} = \int_0^\infty x^{p-1} f(ax)\,dx$, Put $ax = t$

$\tilde{f}(p) = \frac{1}{a^p}\int_0^\infty t^{p-1} f(t)\,dt = \frac{\tilde{f}(p)}{a^p}$

② $\mathcal{M}\{x^a f(x)\} = \tilde{f}(p+a)$

③ $\mathcal{M}\{f(x^a)\} = \frac{1}{a}\tilde{f}(\frac{p}{a})$

④ $\mathcal{M}\{\frac{1}{x} f(\frac{1}{x})\} = \tilde{f}(1-p)$

⑤ $\mathcal{M}\{\log x\, f(x)\} = \frac{d}{dp}\tilde{f}(p)$

This can be shown using $\frac{d}{dp}x^{p-1} = (\log x)\,x^{p-1}$.

⑥ $\mathcal{M}\{f'(x)\} = -(p-1)\tilde{f}(p-1)$, $x^{p-1}f(x) \to 0$ as $x \to 0, \infty$

$\mathcal{M}\{f''(x)\} = (p-1)(p-2)\tilde{f}(p-2)$.

Applications

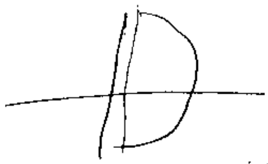
Example 1 $x^2 u_{xx} + x u_x + u_{yy} = 0, \quad 0 \le x < \infty, \; 0 < y < 1$

$$u(x,0) = 0, \qquad u(x,1) = \begin{cases} A, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

A is a constant

Applying the Mellin transform of $u(x,y)$ with respect to x defined by

$$\tilde{u}(p,y) = \int_0^\infty x^{p-1} u(x,y)\, dx$$



$0 < Re(p) < \pi$

The given problem yields

$$\tilde{u}_{yy} + p^2 \tilde{u} = 0, \quad 0 < y < 1,$$

$$\tilde{u}(p,0) = 0, \qquad \tilde{u}(p,1) = A \int_0^1 x^{p-1}\, dx = \frac{A}{p}$$

The solution is given by

$$\tilde{u}(p,y) = \frac{A}{p} \frac{\sin py}{\sin p}, \quad 0 < Re\, p < 1$$

The inverse Mellin transform gives

$$u(x,y) = \frac{A}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p}}{p} \frac{\sin py}{\sin p}\, dp$$

It has simple poles at $p = n\pi$, $n = 1, 2, 3, \ldots$

The residues can be evaluated to obtain

$$u(x,y) = \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n x^{-n\pi} \sin n\pi y.$$