Partial Diff Eqns and BVP

Partial Diff Eqn is a relation involving partial derivatives of the unknown (with one or more independ variables). For example

$$F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u) = 0$$

is a partial D.E involving atmost second order partial ~~differential~~ derivative.

For Linear PDE's $F(\alpha u + \beta v) = \alpha F(u) + \beta F($

A linear partial diff eq of second order in $u = u($ has the form

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

where the letters A to G are either constants or functions of the independent variables x and y only

The DE

$$x u_{xx} + y^2 u_{yy} = f(x,y)$$

is linear but $u\, u_{xx} + u_{yy} = f(x,y)$

is non linear.

Boundary Conditions: u or any of its first derivatives prescribed on boundary.

For example

Dirichlet conditions prescribe u on boundary

Neumann " " normal derivative on boundary.

Robin Condition: $hu + \frac{\partial u}{\partial n}$ is given on the ~~problem~~ boundary.

A BVP is <u>linear</u> if its PDE and all of its boundary conditions are linear.

A linear PDE or B.C. in u is homogeneous if <u>each of its terms</u>, is of <u>first degree</u> in the <u>function</u> u and its derivatives.

Homogeneous PDE

$$u_{xx}(x,y) = \frac{1}{c^2} u_{tt}$$

Homogeneous B.C.

$$u(0,y) = u_x(0,y)$$

Non-homogeneous PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} = e^{x}$$

$$(G \neq 0)$$

Non homogeneous BC.

$$u(x,0) = \sin x$$

~~Classification~~

<u>Properly Posed Problems</u>

1- Does a solution exist

2- Is the solution unique

3- Is the solution stable (continuous dependence on data)

<u>Classification</u>: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

$B^2 - 4AC > 0$ Hyperbolic

$B^2 - 4AC = 0$ Parabolic

$B^2 - 4AC < 0$ Elliptic.

(1) $u_{xx} + u_{yy} = 0$ (Laplace Eqn)

$A = 1, \quad B = 0, \quad C = 1$

$B^2 - 4AC = -4 < 0$ Elliptic

(2) $u_{xx} = \frac{1}{c^2} u_{tt}$, c const. (Wave Equation)

$A = 1, \quad B = 0, \quad C = -\frac{1}{c^2}$

$B^2 - 4AC = \frac{4}{c^2} > 0$ Hyperbolic

(3) $u_{xx} = \frac{1}{a^2} u_t$ Heat Eqn

$A = 1, \quad B = C = 0$

$B^2 - 4AC = 0$

Parabolic.

(4) $u_{xx} - x u_{yy} + u = 0$

$B^2 - 4AC = 4x$

Hyperbolic in half plane $x > 0$

elliptic " " $x > 0$

Parabolic on y-axis $(x = 0)$

General Solution for Constant Coeff Eqns

$$A u_{xx} + B u_{xy} + C u_{yy} = 0$$

A, B, C constants.

Put $r = ax + by$
$s = cx + dy$, $ad - bc \neq 0$

a, b, c are constants to be determined

From chain rule

$$u_x = u_r r_x + u_s s_x = a u_r + c u_s$$

$$u_y = u_r r_y + u_s s_y = b u_r + d u_s$$

$$u_{xx} = r_x^2 u_{rr} + 2 r_x s_x u_{rs} + s_x^2 u_{ss} + r_{xx} u_r + s_{xx} u_s$$

$$= a^2 u_{rr} + 2ac u_{rs} + c^2 u_{ss}$$

$$u_{xy} = ab u_{rr} + (ad + bc) u_{rs} + cd u_{ss}$$

$$u_{yy} = b^2 u_{rr} + 2bd u_{rs} + d^2 u_{ss}$$

PDE $\Rightarrow$

$$(Aa^2 + Bab + Cb^2) u_{rr} + (Ac^2 + Bcd + Cd^2) u_{ss} + [2Aac + B(ad + bc) + 2Cbd] u_{rs} = 0$$

We choose a, b, c, d such that first two terms vanish. For this we assume $A \neq 0$ (Divide by A,

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and $b = d = 1$ and take a ; c to be solutions (5
of $Am^2 + Bm + C = 0$

$$a = m_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$C = m_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

Then we get

$$\left[2 A m_1 m_2 + B(m_1 + m_2) + 2C\right] u_{rs} = 0$$

But

$$m_1 + m_2 = -\frac{B}{A}, \qquad m_1 m_2 = C/A \quad \text{so}$$

Hyperbolic >0
<0 Elliptic

$$\frac{1}{A}\left(\underline{4AC - B^2}\right) u_{rs} = 0$$

It is now clear to see the meaning of $B^2 - 4AC \lessgtr 0$

For hyperbolic/elliptic $B^2 - 4AC \neq 0$

so $u_{rs} = 0$

or $u(r,s) = F(r) + G(s)$

Thus $\boxed{u(x,y) = F(m_1 x + y) + G(m_2 x + y)}$

In parabolic case $B^2 - 4AC = 0$, so we get.

$0 = 0$. $\underline{m_1 = m_2 = m}$

so $r = s$ or $ad - bc = 0$.

In this case we may try

$$\left.\begin{aligned} r &= mx + y \\ s &= x \end{aligned}\right\} \text{ not proportional}$$

a) $a = m,\ b = 1,\ c = 1,\ d = 0$ so that

$$u_{\eta\eta} = 0 \qquad (A \neq 0)$$

$$u(r,s) = F(r) + s\,G(r)$$

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$$u(x,y) = F(mx+y) + x\,G(mx+y)$$

<u>Examples</u>: $u_{xx} + u_{yy} = 0$

① $B^2 - 4AC = -4.$

$Am^2 + Bm + C = 0$ gives

$$m^2 + 1 = 0 \quad, m_1 = i,\ m_2 = -i$$

$$u(x,y) = F(y+ix) + G(y-ix)$$

② $\overset{A}{u_{xx}} - 2\overset{B}{u_{xy}} + \overset{C}{u_{yy}} = 0$

$$B^2 - 4AC = 0$$

$m^2 - 2m + 1 = 0 \quad (a = m,\ b = 1,\ c = 1,\ d = i$

$m_1 = m_2 = 1.$ a

$$u(x,y) = F(x+y) + x\,G(x+y)$$

D'Alembert Solution of Wave Equation

One dimensional wave equation $u_{xx} - c^{-2} u_{tt} = 0 \quad -\infty < x < \infty$

$u(x,0) = f(x),\ u_t(x,0) = g(x)$ (~~Cauchy condition~~

$$B^2 - 4AC = 4c^2 > 0$$

$m_1 = -m_2 = c$ so

$$u(x,t) = F(x+ct) + G(x-ct)$$

Regarding $x+ct = v$, $x - ct = w$ $\quad u(x,t) = F(v) + F(w)$

Using Chain Rule and boundary Condition

$u(x,0) = f(x) \Rightarrow F(x) + G(x) = f(x)$ — (a)

$u_t(x,0) = 0 \Rightarrow aF'(x) - aG'(x) = 0$ — (b)

From (b) $\cancel{a}\, G'(x) = \cancel{a}\, F'(x)$

so $G(x) = F(x) + C$, C constant

From (a) $2F(x) = f(x) - C$

or $F(x) = \frac{1}{2}[f(x) - C]$

$G(x) = \frac{1}{2}[f(x) + C]$

Hence $u(x,t) = \frac{1}{2}[f(x+at) + f(x-a)]$

This is called d' Alembert' solution of the wave equation.

Problems : 1. Use direct integration to solve

(a) $u_{xx} = 6xy$, $0 < x < 1$, $-\infty < y < \infty$

$u(0,y) = y$, $u_x(1,y) = 0$

(b) $u_{xy}(x,y) = 2x$, $x > 0$, $y > 0$

$u(0,y) = 0$, $u(x,0) = x^2$

2) Follow above procedure (d' Alembert Solution) to solve $u_{tt} = a^2 u_{xx}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = 0$

$u_t(x,0) = g(x)$

Separation of Variables :

The method is based upon the assumption that the solution of a homogeneous PDE can be written as product of two functions, each being function of one variable — and product of more function if there are more independent variables.

We shall thus assume

$$u(x,y) = F(x)\,G(y).$$

The full extent of the method will be developed with the help of Fourier series, but we consider the basic idea here.

Example : $y u_x - x u_y = 0$

Put $u(x,y) = F(x)\,G(y)$

then PDE $\Rightarrow$ $y F'(x) G(y) - x F(x) G'(y) = 0$

or $y F'(x) G(y) = x F(x) G'(y)$

Divide by $F(x) G(y)$ both sides

$$y \frac{F'(x)}{F(x)} = x \frac{G'(y)}{G(y)}$$

or $$\frac{1}{x} \frac{F'(x)}{F(x)} = \frac{1}{y} \frac{G'(y)}{G(y)}$$

As L.H.S is a function of one variable (x) alone and R.H.S is a function of y alone, both sides can be equal if both are constant. so

$$\frac{1}{x} \frac{F'(x)}{F(x)} = \frac{1}{y} \frac{G'(y)}{G(y)} = \lambda \text{ (constant)}.$$

This gives two ordinary differential equations

$$\frac{F'(x)}{F(x)} = \lambda x \qquad \text{———} (1)$$

$$\frac{G'(y)}{G(y)} = \lambda y \qquad \text{———} (2)$$

The solutions to (1) and (2) can be easily found to be $F(x) = A e^{\lambda x^2}$ and $G(y) = B e^{\lambda y^2}$

so that $u(x,y) = C e^{\lambda(x^2+y^2)}$, C arbitrary constant.

<u>Examples</u>: (a) $u_x - y u_y = 0$

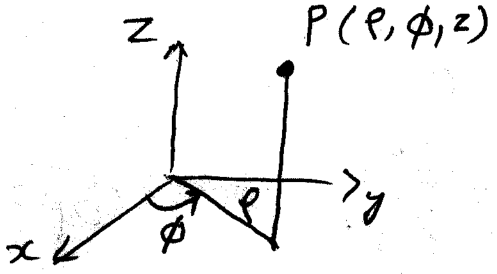
(b) $u_{xy} - u = 0$

<u>Other Coordinate Systems</u>

In many practical problems, the physical situation demands use of cylindrical or spherical coordinates

<u>Cylindrical Coordinates</u>:

$x = \rho \cos\phi$	$\rho = \sqrt{x^2+y^2}$
$y = \rho \sin\phi$	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z$	$z = z$



(Book notation has been followed here)
In other literature (r, θ, z) is used.

By chain Rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{x}{\rho}\frac{\partial u}{\partial \rho} - \frac{y}{\rho^2}\frac{\partial u}{\partial \phi}$$

$$\frac{\partial u}{\partial x} = \cos\phi \frac{\partial u}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial u}{\partial \phi}$$

Repeating this

$$\frac{\partial^2 u}{\partial x^2} = \cos\phi \frac{\partial}{\partial \rho}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial x}\right)$$

Which yields

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\phi \frac{\partial^2 u}{\partial \rho^2} - \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2 u}{\partial \phi \partial \rho} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\sin^2\phi}{\rho}\frac{\partial u}{\partial \rho} + \frac{2\sin\phi\cos\phi}{\rho^2}\frac{\partial u}{\partial \phi}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\phi \frac{\partial^2 u}{\partial \rho^2} + \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2 u}{\partial \phi \partial \rho} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cos^2\phi}{\rho}\frac{\partial u}{\partial \rho} - \frac{2\sin\phi\cos\phi}{\rho^2}\frac{\partial u}{\partial \phi}$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\boxed{\nabla^2 u = \frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2}u_{\phi\phi} + u_{zz}}$$

In two dimensions

$$\rho^2 u_{\rho\rho} + \rho u_\rho + u_{\phi\phi} = 0 \qquad \text{Laplace Equ}$$

$$u_t = k\left(u_{\rho\rho} + \frac{1}{\rho} u_\rho\right) \qquad \text{Heat Equation with polar symmetry.}$$

$$u_{tt} = c^2\left(u_{\rho\rho} + \frac{1}{\rho} u_\rho\right) \qquad \text{Wave Equation with polar symmetry}$$

In case of polar symmetry, u is independent of angle ϕ and so $u(\rho, t)$ only.

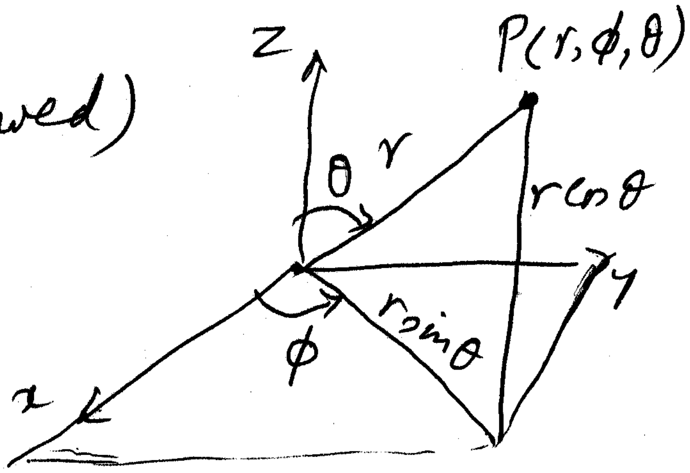
Spherical Coordinates

(Notation of Book followed)

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$



We can check!

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}.$$

See the relevant section of the text book also.