# Decomposition and Factorization Theorems

We have noticed that in applying the Wiener-Hopf procedure, the factorization (product splitting or quotient splitting) and additive decomposition into one sided functions with over-laping ranges of analyticity is crucial. In the example of integral equation considered, this was possible by inspection. In the following we given another example where this can be easily achieved.

<u>Example</u> As an example, let us consider

$$f(\alpha) = \frac{1}{(\alpha - K\cos\theta)(\alpha+K)^{1/2}} \qquad (1)$$

where $K = K_1 + iK_2$, $K_1, K_2 > 0$. Then $f(\alpha)$ has a pole at $\alpha = K\cos\theta$ (pole in the upper half plane) and has a branch point at $\alpha = -K$. However, $(\alpha+K)^{1/2}$ can be made single valued by staying in the upper half-plane. Let us write

$$f(\alpha) = \frac{1}{\alpha - K\cos\theta}\left[\frac{1}{(\alpha+K)^{1/2}} \underbrace{- \frac{1}{(K\cos\theta+K)^{1/2}} + \frac{1}{(K\cos\theta+K)^{1/2}}}\right]$$

Subtracting and adding the term.

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Notice that the term we have added and subtracted is the residue of $f(\alpha)$ at the pole $\alpha = K\cos\theta$. We write

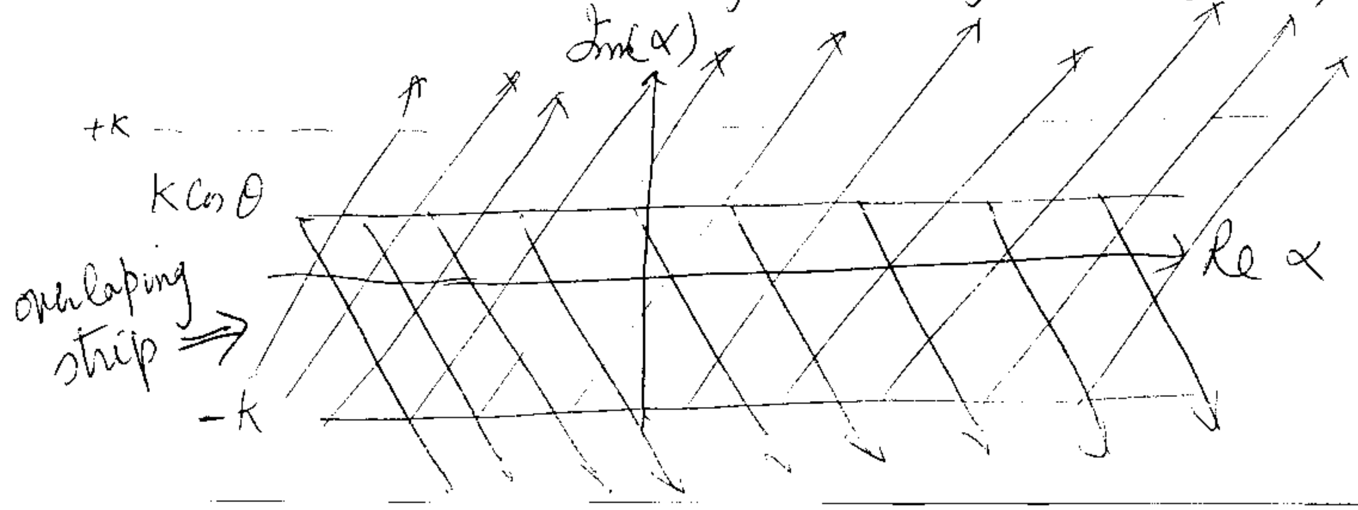
$$f(\alpha) = \frac{1}{(\alpha - K\cos\theta)}\left[\frac{1}{(\alpha+K)^{1/2}} - \frac{1}{(K\cos\theta+K)^{1/2}}\right] + \frac{1}{(\alpha - K\cos\theta)(K\cos\theta+K)^{1/2}} \qquad (2)$$

In equation (2) the first term

$$\frac{1}{(\alpha - K\cos\theta)}\left[\frac{1}{(\alpha+K)^{1/2}} - \frac{1}{(K\cos\theta+K)^{1/2}}\right]$$

is no longer singular at $\alpha = K\cos\theta$ (the bracketed term $\to 0$ as $\alpha \to K\cos\theta$).

~~Thes~~ This term is thus analytic in the upper half plane $\text{Im}(\alpha) > -K$. We call it $f_+(\alpha)$. The second term has a pole at $\alpha = K\cos\theta$ and so analytic in the lower half-plane $\text{Im}(\alpha) < K\cos\theta$. We call it $f_-(\alpha)$. Thus $f(\alpha) = f_+(\alpha) + f_-(\alpha)$



General Theorems.

**Decomposition:** Let $f(\alpha)$ be such that it is analytic in the strip $d < \operatorname{Im}\alpha < C$ and $f(\alpha) \to 0$ as $|\operatorname{Re}\alpha| \to \infty$

Then we can write $f(\alpha) = f_+(\alpha) + f_-(\alpha)$

where

$$f_\pm(\alpha) = \frac{\pm 1}{2\pi i}\int_{-\infty + iC \atop -\infty + id}^{\infty + id \atop \infty + iC} \frac{f(\omega)}{\omega - \alpha}\, d\omega$$

Example: Let us apply this theorem to $f(\alpha)$ considered above

$$f(\alpha) = \frac{1}{(\alpha - K\cos\theta)(\alpha + K)^{1/2}}$$

By the above theorem (check the conditions are satisfied)

$$f_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty + ic}^{\infty + ic} \frac{d\omega}{(\omega - K\cos\theta)(\omega + K)^{1/2}(\omega - \alpha)}$$

enclose the contour as a semi-circle in the upper half plane and line $\operatorname{Im}(\omega) = C$. Then $\omega = K\cos\theta$ and $\omega = \alpha$ are the two poles inside this contour.

$R_1 =$ Residue at $\omega = K\cos\theta = \dfrac{1}{(K\cos\theta + K)^{1/2}(K\cos\theta - \alpha)}$

$R_2 =$ Res at $\omega = \alpha = \dfrac{1}{(\alpha - K\cos\theta)(\alpha + K)^{1/2}}$

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$$f_+(\alpha) = 2\pi i\left(\frac{1}{2\pi i}\text{ Sum of Residues}\right)$$

$$= \frac{1}{\alpha - K\cos\theta}\left[\frac{1}{(\alpha+K)^{1/2}} - \frac{1}{(K\cos\theta + K)^{1/2}}\right]$$

as before.

<u>Try finding</u> $f_-(\alpha)$.

<u>Factorization</u>: We can apply the above theorem on $\log f(\alpha)$ now. $\log f(\alpha)$ will now have to satisfy requirements of the theorem implying $f(\alpha) \to 1$ as $Re|\alpha| \to \infty$

Then $\log[f(\alpha)] = g(\alpha) = g_+(\alpha) + g_-(\alpha)$

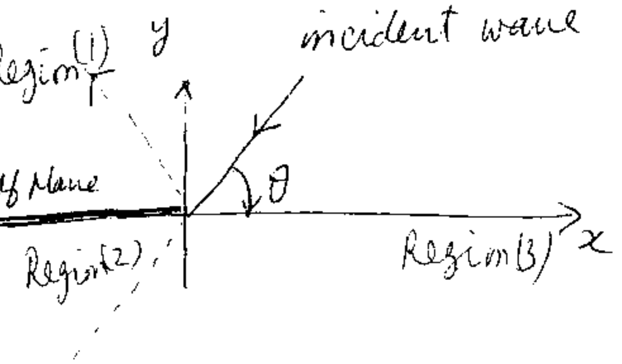
$g_\pm(\alpha)$ given by above theorem.

Then $f(\alpha) = e^{g_+(\alpha)}\, e^{g_-(\alpha)}$ (Factors)

or $= \dfrac{e^{g_+(\alpha)}}{e^{-g_-(\alpha)}}$ (Quotient).

Mixed Boundary value Problem :

Wave Equation in two dimension




$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \quad -(1)'$$

Assume wave to be time harmonic,

$$\phi = \varphi(x,y)\, e^{-i\omega t} \quad , \omega \text{ angular frequency.}$$

$$(1)' \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + K^2 \phi = 0 \quad \text{———} (1)$$

where $K^2 = \frac{\omega^2}{c^2}$ (wave number) is assumed to be complex with a positive imaginary part.

We assume the incident wave $\phi_i$ as

$$\phi_i = \exp\{-ikx\cos\theta - iky\sin\theta\} \quad , \quad 0<\theta<\pi \quad \text{———} (2)$$

The plane $x<0$ has a rigid plane (on which derivative of $\phi$ vanishes). [Other kind of plane can also be assumed].

Due to plane, we get the 'diffracted' field. The total field consists of incident + diffracted field. Thus

$$\phi_t = \phi + \phi_i$$

The total as well as incident fields are wave forms so satisfy wave equation (1).

Thus $\phi$ also satisfies (2), i.e.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0. \quad \text{—— (3)}$$

<u>Boundary Conditions</u>.

(i) $\frac{\partial \phi_t}{\partial y} = 0$ on $y = 0$, $-\infty < x \leq 0$, so that

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \phi_i}{\partial y} = ik \sin\theta \exp(-ikx\cos\theta) \text{ on}$$
$$y = 0, \quad -\infty < x \leq 0$$

(ii) $\frac{\partial \phi_t}{\partial y}$ and therefore $\frac{\partial \phi}{\partial y}$ are continuous on $y = 0$

$-\infty < x < \infty$ i.e.

$$\frac{\partial \phi}{\partial y}(x, 0+) = \frac{\partial \phi}{\partial y}(x, 0-), \quad -\infty < x < \infty.$$

(iii) $\phi_t$ and therefore $\phi$ are continuous on $y = 0$,
$0 < x < \infty$ i.e.

$$\phi(x, 0+) = \phi(x, 0-), \quad 0 < x < \infty.$$

$(x, y)$ plane can be divided in three regions

(1) Region (1) in which $\phi$ consists of a diffracted wave and a reflected wave

(2) Region (2) in which $\phi$ consists of a diffracted wave <u>minus</u> an incident wave (shadow region)

(3) Region (3) in which only a diffracted wave is present.

As $r \to \infty$, diffracted wave is viewed as being produced by the diffracting plane acting as a plane

Thus diffracted wave has behaviour

$$\lim_{r\to\infty} C_1 H_0^{(1)}(kr) \sim C_2 r^{-1/2} e^{ik_1 r} e^{-k_2 r}$$

$$(k = k_1 + i k_2) \qquad C_1, C_2 \text{ Constants.}$$

In region (1) the reflected wave is given by $\exp(-ikx\cos\theta + iky\sin\theta)$ (Notice the resolution of components along the axes)

Thus we have

(IV) For any fixed $y$, $y \geq 0$ or $y \leq 0$

(a) $|\phi| < C_3 \exp(k_2 x\cos\theta - k_2|y|\sin\theta)$
$-\infty < x < -|y|\cot\theta$

(b) $|\phi| < C_4 \exp\{-k_2(x^2+y^2)^{1/2}\}$ for $-|y|\cot\theta < x < \infty$.

There are some edge conditions to be satisfied to ensure uniqueness.

<u>Fourier Transforms</u>: Let us introduce the Fourier transforms

$$\phi^*(\alpha, y) = \phi^*_+(\alpha, y) + \phi^*_-(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi\, e^{i\alpha x}\, dx \quad , \alpha = \sigma + i\tau$$

$$\phi^*_+(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \phi\, e^{i\alpha x}\, dx \; ; \; \phi^*_-(\alpha, y) = \int_{-\infty}^{0} \phi\, e^{i\alpha x}\, dx$$

From the above conditions we have $|\phi| < D_1 \exp(-k_2 x)$ as $x \to \infty$ and $|\phi| < D_2 \exp(k_2 \cos\theta\, x)$ as $x \to -\infty$ where $D_1, D_2$ are constants

Therefore $\phi^*_+$ is analytic for $\tau > -k_2$

$\phi^*_-$ is analytic for $\tau < k_2\cos\theta$

and so $\phi^*$ is analytic in strip $-k_2 < \tau < k_2\cos\theta$.

If we now apply Fourier transform to (3), we get

$$\frac{d^2\phi^*(x,y)}{dy^2} - \gamma^2\phi^*(\alpha,y) = 0 \ , \ \gamma = (\alpha^2-k^2)^{1/2}$$

The solution is given by

$$\phi^*(x,y) = \begin{cases} A_1(\alpha)e^{-\gamma y} + B_1(\alpha)\, e^{\gamma y} & ; \ y \geq 0 \\ A_2(\alpha)e^{-\gamma y} + B_2(\alpha)\, e^{\gamma y} & ; \ y \leq 0 \end{cases}$$

The real part of $\gamma$ is positive in $k_2 < \tau < k_2$

Using the transformed boundary conditions
(Take Fourier transform and see)

$B_1 = A_2 = 0$ and $A_1(\alpha) = -B_2(\alpha) = A(\alpha)$, say

$$\phi^*(\alpha,y) = \begin{cases} A(\alpha)\, e^{-\gamma y} & ; \ y \geq 0 \\ -A(\alpha)\, e^{\gamma y} & ; \ y \leq 0 \end{cases} \quad (4)$$

(We may write $\phi^*(y)$ instead of $\phi^*(\alpha,y)$ for brevity)

$\phi_\pm^*(\alpha,y)$ for $y=0$ is written as $\phi_\pm^*(0)$ if it is not continuous across $y=0$ (as $\phi$ may not be)

$$\phi_-^*(\pm 0) = \phi_-(\alpha, \pm 0) = \lim_{y\to\pm 0} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \phi\, e^{i\alpha x}\, dx .$$

Now because of continuity of $\phi$ for $x>0$ (B.C (iii))

$$\phi_+^*(+0) = \phi_+^*(-0) = \phi_+^*(0) \quad (\text{say}) \quad \text{——} \ (5)$$

$\phi_+^{*\prime}(\alpha,y)$, $\phi_-^{*\prime}(\alpha,y)$ are defined are respective half range transforms of $\phi(x,y)$. Using these From B.C (ii)

$$\left.\begin{aligned} \phi_+^{*\prime}(\alpha,+0) &= \phi_+^{*\prime}(\alpha,-0) = \phi_+^{*\prime}(0) \\ \text{similarly } \ \phi_-^{*\prime}(\alpha,+0) &= \phi_-^{*\prime}(\alpha,-0) = \phi_-^{*\prime}(0) \end{aligned}\right\} (6)$$

On applying these, we can write (4) as



$$\left.\begin{aligned}
&\phi_+^*(0) + \phi_-^*(+0) = A(\alpha) \quad (a)\\
&\phi_+^*(0) + \phi_-^*(-0) = -A(\alpha) \quad (b)\\
&\text{Derivative would lead to}\\
&\phi_+^{*\prime}(0) + \phi_-^{*\prime}(0) = -\gamma\, A(\alpha) \quad (c)
\end{aligned}\right\} \quad (7)$$

( Derivative of $\phi$ is continuous for $-\infty < x < \infty$ )

(a) and (b) are analogous. Adding these

$$2\phi_+^*(0) = -\phi_-^*(+0) - \phi_-^*(-0). \qquad (8)$$

Now subtract (b) from (a)

$$\phi_-^*(+0) - \phi_-^*(-0) = 2A(\alpha) \qquad \text{—— } (9)$$

Eliminate $A(\alpha)$ from and (9) and (7 c)

$$\underbrace{\phi_+^{*\prime}(0) + \phi_-^{*\prime}(0)}\; = -\tfrac{1}{2}\gamma \left\{\phi_-^*(+0) - \phi_-^*(-0)\right\} \quad \text{—} (10)a)$$

Notice that $\phi_-^{*\prime}(0) = \mathcal{F}_-\left\{\frac{\partial\phi}{\partial y}\right\}_{\substack{y=0\\ x<0}}$ is known

through boundary condition (i) as

$$\phi_-^{*\prime}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{i\alpha x}\left(ik\sin\theta\, e^{-ikx\cos\theta}\right)dx = \frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha - k\cos\theta)} \qquad (10b)$$

For simplicity, we write

$$\phi_-^*(+0) - \phi_-^*(-0) = 2D_- \;;\; \phi_-^*(+0) + \phi_-^*(-0) = 2S_-$$

$D_-$, $S_-$ are regular in $\tau < k_2\cos\theta$.

Equations (8) and (10) can be written as

$$\phi_+^*(0) = -S_- \longrightarrow (\text{This is not known}) \qquad (11)$$

$$\phi_+^{\prime *}(0) + \frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha - k\cos\theta)} = -\gamma D_- \uparrow \qquad \text{—}(12)$$

$\phi_+^*$, $\phi_+^{*\prime}$, $S_-$, $D_-$ not known.

Each equation is valid in the strip
$-K_2 < \tau < K_2 \cos\theta$

Solution of the Problem.

In equation (12), $\gamma = (\alpha^2 - K^2)^{1/2} = (\alpha+K)^{1/2}(\alpha-K)^{1/2}$

Branch of each is defined so that $(\alpha \pm K)^{1/2} \to \alpha^{1/2}$ as $\sigma \to +\infty$ in the strip $-K_2 < \tau < K_2$. $(\alpha+K)^{1/2}$ is regular and non zero in $\tau > -K_2$, so Divide by it

$$\frac{\phi_+^{*\prime}(0)}{(\alpha+K)^{1/2}} + \frac{K\sin\theta}{\sqrt{2\pi}\,(\alpha+K)^{1/2}(\alpha - K\cos\theta)} = -(\alpha-K)^{1/2} D_-$$

The second term on L.H.S is a mixed term and we have discussed its factorization as

$$\frac{K\sin\theta}{\sqrt{2\pi}\,(\alpha+K)^{1/2}(\alpha-K\cos\theta)} = \frac{K\sin\theta}{(2\pi)^{1/2}(\alpha-K\cos\theta)}\left\{\frac{1}{(\alpha+K)^{1/2}} - \frac{1}{(K+K\cos\theta)^{1/2}}\right\}$$

$$+ \frac{K\sin\theta}{\sqrt{2\pi}\,(K+K\cos\theta)^{1/2}(\alpha-K\cos\theta)}$$

$$= H_+(\alpha) + H_-(\alpha) \qquad \text{(say)}$$

where $H_+(\alpha)$ is analytic in $\tau > -K_2$

$H_-(\alpha)$ is so in $\tau < K_2\cos\theta$

We thus have

$$J(\alpha) = (\alpha+K)^{-1/2}\phi_+^*(0) + H_+(\alpha) = -(\alpha-K)^{1/2} D_- - H_-(\alpha)$$

By analytic continuation $J(\alpha)$ is regular in the whole of $\alpha$-plane and so an entire function

Under certain conditions, one can usually show that $J(\alpha) = 0$.

Thus

$$\left.\begin{aligned} \phi_+^{*\prime}(0) &= -(\alpha+k)^{1/2} H_+(\alpha) \quad (a) \\ D_- &= -(\alpha-k)^{-1/2} H_-(\alpha) \quad -(b) \end{aligned}\right\} (13)$$

using (7) C, (10 b) and 13(a) we can find the unknown, $A(\alpha)$ as

$$A(\alpha) = -\frac{1}{(2\pi)^{1/2}} \frac{k \sin\theta}{(k+k\cos\theta)^{1/2}(\alpha+k)^{1/2}(\alpha-k\cos\theta)}$$

We can insert this value in the solution (4) and take inverse Fourier transform to obtain

$$\phi = \mp \frac{1}{2\pi}(k-k\cos\theta)^{1/2} \int_{-\infty+ia}^{\infty+ia} \frac{e^{-i\alpha x \mp \gamma y}}{(\alpha-k)^{1/2}(\alpha-k\cos\theta)} \, d\alpha$$

This completes the Jones' method.