Hankel transform.

The Hankel transform involves Bessel functions and is more suitable in problems with cylindrical symmetry. We can 'derive' Hankel transform definition from the Fourier transform definition but it is left to the reader.

Hankel transform of order n (called so as it involves the Bessel function of order n) is defined as

$$\mathcal{H}_n\{f(r)\} = \tilde{f}_n(\alpha) = \int_0^\infty r\, J_n(\alpha r) f(r)\, dr,$$

the inversion is given by

$$\mathcal{H}_n^{-1}\{\tilde{f}_n(r)\} = f(r) = \int_0^\infty \alpha\, J_n(\alpha r)\, \tilde{f}_n(\alpha)\, d\alpha.$$

We may skip subscript n and write $\tilde{f}(\alpha)$ whenever there is no fear of confusion.

It follows that $\mathcal{H}_n^2\{f(r)\} = f(r)$

Properties: ① If $\mathcal{H}_n\{f(r)\} = \tilde{f}_n(\alpha)$ then

$$\mathcal{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\alpha}{a}\right), \quad a>0$$

$\mathcal{H}_n\{f(ar)\} = \int_0^\infty r\, J_n(\alpha r) f(ar)\, dr$. Put $ar = s$, $dr = \frac{ds}{a}$

so $\mathcal{H}_n\{f(ar)\} = \frac{1}{a^2}\int_0^\infty s\, J_n\left(\frac{\alpha}{a} s\right) f(s)\, ds = \frac{1}{a^2}\tilde{f}_n\left(\frac{\alpha}{a}\right).$

② $\mathcal{H}_n\{f'(r)\} = \frac{\alpha}{2n}\left[(n-1)\tilde{f}_{n+1}(\alpha) - (n+1)\tilde{f}_{n-1}(\alpha)\right\}$ $n \geq 1$

$\mathcal{H}_1\{f'(r)\} = -\alpha\, \tilde{f}_0(\alpha)$

Proof: $\mathcal{H}_n\{f'(r)\} = \int_0^\infty r\, J_n(\alpha r) f'(r)\, dr$

$$= \left[r f(r) J_n(\alpha r)\right]_0^\infty - \int_0^\infty f(r)\frac{d}{dr}\left[r J_n(\alpha r)\right] dr$$

using $\frac{d}{dr}\left[r J_n(\alpha r)\right] = J_n(\alpha r) + r\alpha J_n'(\alpha r)$

$$= J_n(\alpha r) + r\alpha J_{n-1}(\alpha r) - n J_n(\alpha r)$$

$$= (1-n) J_n(\alpha r) + r\alpha J_{n-1}(\alpha r),$$

and the fact that $r f(r) \to 0$ as $r \to 0$ & $r \to \infty$,

we obtain

$$\mathcal{H}_n\{f'(r)\} = (n-1)\int_0^\infty f(r) J_n(\alpha r)\,dr - \alpha \tilde{f}_{n-1}(\alpha)$$

Now $$J_n(\alpha r) = \frac{r\alpha}{2n}\left[J_{n-1}(\alpha r) + J_{n+1}(\alpha r)\right]$$

(Recurrence Relation)

Therefore,

$$\mathcal{H}_n\{f'(r)\} = -\alpha \tilde{f}_{n-1}(\alpha) + \alpha\left(\frac{n-1}{2n}\right)\int_0^\infty r f(r)\left[J_{n-1}(\alpha r) + J_{n+1}(\alpha r)\,dr\right]$$

$$= -\alpha \hat{f}_{n-1}(\alpha) + \alpha\left(\frac{n-1}{2n}\right)\left[\tilde{f}_{n-1}(\alpha) + \tilde{f}_{n+1}(\alpha)\right]$$

$$= \frac{\alpha}{2n}\left[(n-1)\hat{f}_{n+1}(\alpha) - (n+1)\check{f}_{n-1}(\alpha)\right].$$

$n=1$ gives

$$\mathcal{H}_1\{f'(r)\} = -\alpha \hat{f}_0(\alpha).$$

(3) $$\mathcal{H}_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2 f}{r^2}\right\} = -\alpha^2 \tilde{f}_n(\alpha)$$

if $r f'(r)$ and $r f(r) \to 0$ as $r \to 0$ and $r \to \infty$.

Proof $$\text{L.H.S} = \int_0^\infty J_n(\alpha r)\left[\frac{d}{dr}\left(r\frac{df}{dr}\right)\right]dr - \int_0^\infty \frac{n^2}{r^2}\left[r J_n(\alpha r)\right] f(r)\,dr$$

Using integration by parts

$$= \underbrace{\left[\left(r\frac{df}{dr}\right) J_n(\alpha r)\right]_0^\infty}_{\to 0} - \alpha\int_0^\infty r\frac{df}{dr} J_n'(\alpha r)\,dr - \int_0^\infty \frac{n^2}{r^2}\left[r J_n(\alpha r)\right] * f(r)\,dr$$

$$= -\underbrace{\left[\alpha r f(r) J_n'(\alpha r)\right]_0^\infty}_{\to 0} + \int_0^\infty \frac{d}{dr}\left[\alpha r J_n'(\alpha r)\right] f(r)\,dr - \int_0^\infty \frac{n^2}{r^2}\left[r J_n(\alpha r)\right] f(r)\,dr.$$

As $J_n(\alpha r)$ satisfies the Bessel equation

$$\frac{d}{dr}\left[\alpha r J_n'(\alpha r)\right] + r\left[\alpha^2 - \frac{n^2}{r^2}\right] J_n(\alpha r) = 0, \text{ we get}$$

$$H_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}\right\} = -\int_0^\infty \left(\alpha^2 - \frac{n^2}{r^2}\right) r f(r) J_n(\alpha r)\, dr$$

$$- \int_0^\infty \frac{n^2}{r^2}\left[r f(r)\right] J_n(\alpha r)\, dr$$

$$= -\alpha^2 \int_0^\infty r J_n(\alpha r) f(r)\, dr = -\alpha^2 H_n[f(r)] = -\alpha^2 \tilde{f}_n(\alpha)$$

By putting $n=0$, we get

$$H_0\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right\} = -\alpha^2 \tilde{f}_0(\alpha)$$

$n=1$ gives

$$H_1\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{1}{r^2} f(r)\right\} = -\alpha^2 \tilde{f}_1(\alpha).$$

These results are widely used in problems with cylindrical symmetry.

Applications

Example 1 (Integral Equation)

$$\phi(x) - \lambda \int_0^\infty y J_n(xy)\, \phi(y)\, dy = f(x)$$

This can be written as $\phi - \lambda H_n(\phi) = f$

$$n \; \cancel{\phi = f + \lambda H_n(f)}$$

Taking H_n $\quad \cancel{H_n\phi = H_n f + \lambda f}$ $\Big]$ $H_n\phi = \dfrac{\phi - f}{\lambda}$

$$H_n \phi - \lambda \phi = H_n f \Rightarrow \frac{\phi - f}{\lambda} - \lambda\phi = H_n f$$

$$\Rightarrow \frac{\phi - f - \lambda^2\phi}{\lambda} = H f$$

$$\Rightarrow \phi = \frac{f + \lambda H f}{1 - \lambda^2}$$

Example 2 (PDE)

Consider the free vibration of a large circular elastic membrane governed by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0 < r < \infty, \; t > 0$$

$$u(r,0) = f(r); \quad u_t(r,0) = g(r), \quad 0 \le r < \infty.$$

We use zeroth order Hankel transform w.r.t. r

$$\hat{u}(\alpha,t) = \int_0^\infty r\, J_0(\alpha r)\, u(r,t)\, dr.$$

$$\text{PDE} \Longrightarrow \frac{d^2 \hat{u}}{dt^2} + c^2 \alpha^2 \hat{u} = 0$$

The initial conditions become

$$\hat{u}(\alpha,0) = \hat{f}(\alpha), \quad \hat{u}_t(\alpha,0) = \hat{g}(\alpha)$$

The general solution is given by

$$\hat{u}(\alpha,t) = \hat{f}(\alpha)\cos(\alpha c t) + \frac{\hat{g}(\alpha)}{\alpha c}\sin(\alpha c t)$$

The inverse Hankel transform leads to

$$u(r,t) = \int_0^\infty \alpha\, \hat{f}(\alpha)\cos(\alpha c t)\, J_0(\alpha r)\, d\alpha + \int_0^\infty \frac{1}{c}\hat{g}(\alpha)\sin(\alpha c t)\, J(\alpha r)\, d\alpha.$$

Example 3 (Steady State Temperature)

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = -q(r), \quad 0 < r < \infty, \; 0 < z < \infty.$$

$$u(r,0) = 0, \quad 0 < r < \infty.$$

Applying zeroth order Hankel transform

$$\frac{d^2 \hat{u}}{dz^2} - \alpha^2 \hat{u} = -\hat{q}(\alpha), \quad \hat{u}(\alpha,0) = 0$$

The solution: $\hat{u}(\alpha,z) = A e^{-\alpha z} + \frac{1}{\alpha^2}\hat{q}(\alpha)$

We find $A = -\frac{1}{\alpha^2}\hat{q}(\alpha)$ and so

$$u(r,z) = \int_0^\infty \frac{\hat{q}(\alpha)}{\alpha}\left(1 - e^{-\alpha z}\right) J_0(\alpha r)\, d\alpha.$$

Example 4 Biharmonic Eqn in Axi symmetric domain

$$\nabla^4 u(r,z) = 0, \quad 0 \le r < \infty, \; z > 0$$
$$u(r,0) = f(r), \quad 0 \le r < \infty$$
$$\frac{\partial u}{\partial z} = 0, \quad \text{on } z = 0, \quad 0 \le r < \infty$$
$$u(r,z) \to 0 \quad \text{as } r \to \infty.$$

Here $\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$

Taking Hankel transform of order zero

$\mathcal{H}_0(u(r,z)) = \hat{u}(\alpha,z)$ we get

$$\left(\frac{d^2}{dz^2} - \alpha^2\right)^2 \hat{u}(\alpha,z) = 0, \quad z > 0$$

The auxiliary equation has repeated roots $\pm\alpha, \pm\alpha$

Thus, $\hat{u}(\alpha,z) = (A + Bz)\, e^{-\alpha z}$, where

$e^{\alpha z}$ term has been neglected to satisfy ~~[illegible]~~ boundedness conditions.

$$\mathcal{H}_0\{u(r,0)\} = \mathcal{H}_0\{f(r)\} \Rightarrow \hat{u}(\alpha,0) = \tilde{f}(\alpha)$$

$$\mathcal{H}_0\left(\frac{\partial u}{\partial z} = 0\right\} \Rightarrow \frac{d\hat{u}(\alpha,0)}{dz} = 0.$$

Now $\dfrac{d\hat{u}(\alpha,z)}{dz} = -\alpha(A + Bz)\, e^{-\alpha z} + B\, e^{-\alpha z}$

$$\left.\frac{d\hat{u}}{dz}\right|_{z=0} = 0 \Rightarrow -\alpha A = -B \Rightarrow \alpha A = B$$

So $\hat{u}(\alpha,z) = A(1 + \alpha z)\, e^{-\alpha z} + \alpha z A e^{-\alpha z}$

$$\tilde{u}(\alpha,0) = \tilde{f}(\alpha) \Rightarrow \quad A = \tilde{f}(\alpha),$$
$$\hat{u}(\alpha,z) = \tilde{f}(\alpha)(1 + \alpha z)\, e^{-\alpha z}$$

The inverse Hankel transform gives

$$u(r,z) = \int_0^\infty (1+\alpha z)\, \tilde{f}(\alpha)\, J_0(\alpha r)\, e^{-\alpha z}\, d\alpha.$$