Mappings in Complex Plane

Let f be a mapping from z-plane to w plane:

$f: D \to D^*$

Angle Preserving: We say that f preserves angles if for any z_0 in D, any two smooth curves in D intersecting at an angle θ at z_0 have images that intersect at z_0 at the same angle in D^*.

Orientation Preserving: We say that f preserves orientation if counterclockwise rotation in D is mapped into counterclockwise rotation in D^*

Conformal Mapping: If f preserves angle as well as orientation, it is known as conformal mapping.

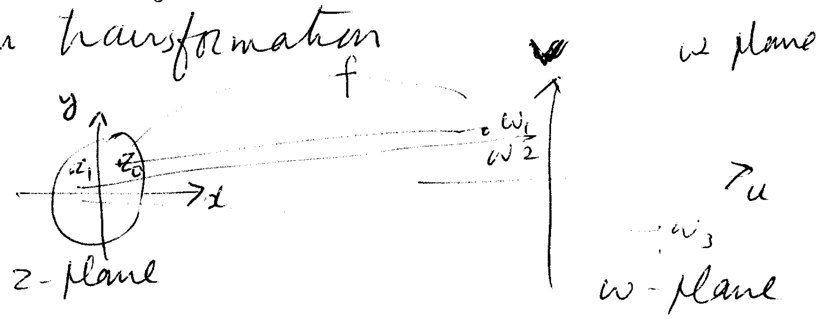
Conformal Mappings & Applications

In many engineering and physical problems one is interested in solving the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in some region } D \text{ of } \mathbb{C}. \quad \text{—} (1)$$

$\phi(x,y)$ is also required to satisfy certain boundary conditions. We know that real and imaginary parts of an analytic function satisfy (1) (harmonic functions). Thus solving a boundary value problem involving (1) reduces to finding an analytic function in D that satisfies the required boundary conditions on the boundary ∂D of D. It turns out that this can be easily done if D is either the upper half plane or the unit disk. Thus one should first perform a change of variables from the complex variable z to the complex variable w given by $w = f(z)$, such that the region D is mapped to the upper half of the w plane. Generally speaking, such transformations are called conformal mappings.

Thus $w = f(z) = u(x,y) + i\,v(x,y)$ is a planar transformation



If $z(t) = x(t) + i y(t)$, $a \le t \le b$ describes a curve C in the region, then $w = f(z(t))$, $a \le t \le b$ is a parametric representation of the corresponding curve C' in the w-plane. A point z on the level curve $u(x,y) = a$ will be mapped to a point w that lies on $u = a$, and a point z on the level curve $v(x,y) = b$ will be mapped to a point w that lies on the horizontal line $v = b$.

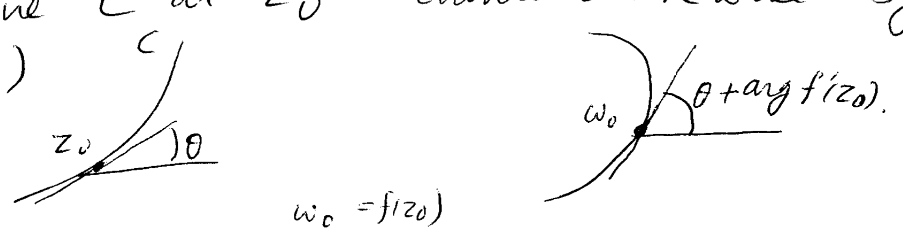
~~Mapping Theorem~~ Riemann Theorem: If D is a simply connected region, which is not the entire complex z-plane, then there exists an analytic function $f(z)$ such that $w = f(z)$ transforms D onto the upper half w plane.

Remark: This theorem does not give a method of finding such a $f(z)$.

~~Angle Pres~~ Rotation Property.

Let C be a curve in the complex z plane. Let $w = f(z)$ where $f(z)$ is some analytic function of z defining a change of variables from complex z-plane to the w-plane. Under this transformation, the curve C is mapped to some curve C' in w plane:

Let z_0 be a point on the curve C and assume $f'(z_0) \neq 0$. Then $w = f(z)$ rotates the tangent to the curve C at z_0 counterclockwise by $\arg f'(z_0)$



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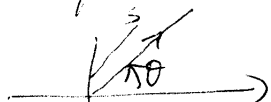
Proof : Points on a continuous curve C are characterized by parametric representation $x = x(s)$, $y = y(s)$, where $x(s)$ and $y(s)$ are real differentiable functions of real parameter s.

[For example a straight line  is given by $\left.\begin{array}{l} x = s\cos\theta \\ y = s\sin\theta \end{array}\right\}$ θ fixed; s parameter.

and a circle with center O and radius R is given by $\left.\begin{array}{l} x = R\cos s \\ y = R\sin s \end{array}\right\}$ s parameter]

Thus $C: \; Z(s) = x(s) + i\,y(s)$

Suppose $f(z)$ is analytic for z in some domain of the complex z plane denoted by D. Let us confine our self to arc C (part of C contained in D)

$$C: \; Z(s) = x(s) + i\,y(s), \quad s \in [a,b] \; \text{i.e.} \; a \le s \le b. \qquad (2)$$

If we write $w = u(x,y) + i\,v(x,y)$, $u, v \in \mathbb{R}$,

image of (2) is arc C' given by

$$w(s) = u(x(s), y(s)) + i\,v(x(s), y(s))$$

As x and y depend continuously on s, it follows that u and v also are continuous functions of x, y. Similarly image of a differentiable arc is also differentiable. However image of an non intersecting arc may be having an intersection with itself. In fact if $f(z_1) = f(z_2)$, $z_1, z_2 \in D$, then any arc C passing through z_1, z_2 will ~~als~~ have its image intersect itself.

Let us define $\dfrac{dz(s)}{ds} = \dfrac{dx(s)}{ds} + i\,\dfrac{dy(s)}{ds}$

Let $f(z)$ be analytic in a domain containing open NHD of $z_0 \equiv s(s_0)$. The image of C is $w(s) = f(z(s))$

thus $$\left.\frac{dw(s)}{ds}\right|_{s=s_0} = f'(z_0)\left.\frac{dz(s)}{ds}\right|_{s=s_0}$$

If $f'(z_0) \neq 0$ and $z'(s_0) \neq 0$, it follows that $w'(s_0) \neq 0$ and

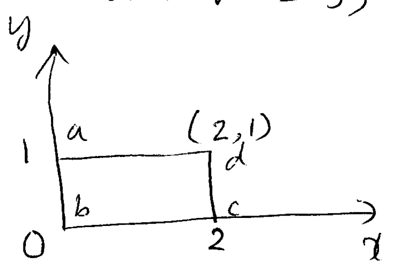
$$\boxed{\arg(w'(s_0)) = \arg(z'(s_0)) + \arg f'(z_0)}$$

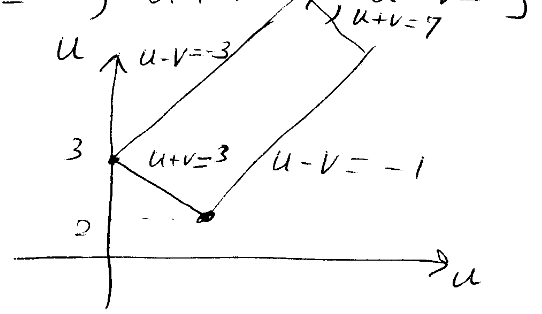
Thus we get a rotation by an amount of $\arg f'(z_0)$. This proves our assertion

Example. D: rectangular region in z plane bounded by $x=0$, $y=0$, $x=2$, $y=1$.

Consider $w = f(z) = (1+i)z + (1+2i)$ which gives the rectangular region D' of the w-plane bounded by

$$u+v=3,\quad u-v=-1,\quad u+v=7,\quad u-v=-3$$





Check that a, b, c and d are mapped onto $(0,3)$, $(1,2)$, $(3,4)$ and $(2,5)$ respectively

Rotation is by $\arg(f'(z_0)) = \arg(1+i)$ which is $\pi/4$.

Example The mapping $w = e^z$ is conformal throughout the entire plane since $\frac{d}{dz}(e^z) \neq 0$ for each z.

Two lines $x = c_1$ and $y = c_2$ in the z plane are mapped onto a positively oriented circle centered at origin and a line passing through the origin as following example shows.

Example: The mapping $w = f(z) = \frac{1}{z}$

$\frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$ Thus $z = 0$ is not in the domain

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = -\frac{y}{x^2+y^2}$$

When $c_1 \neq 0$, $u(x,y) = c_1$ gives

$$c_1 = \frac{x}{x^2+y^2} \Rightarrow c_1(x^2+y^2) - x = 0$$

$$\Rightarrow x^2+y^2 - \frac{1}{c_1}x = 0$$

$$\Rightarrow \left(x - \frac{1}{2c_1}\right)^2 + y^2 = \left(\frac{1}{2c_1}\right)^2$$

Circle of radius $\frac{1}{2c_1}$, center at $\left(\frac{1}{2c_1}, 0\right)$

Similarly $v(x,y) = c_2$ ($c_2 \neq 0$) gives

$$x^2 + \left(y + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_1}\right)^2$$

Since $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$ Thus

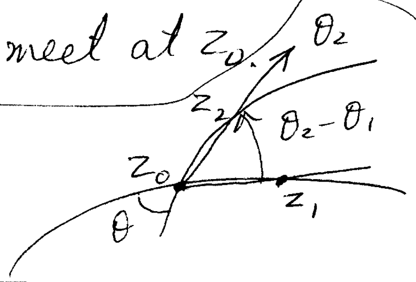
$$f^{-1}(w) = \frac{1}{w} \text{ so } f = f^{-1}$$

We can therefore conclude that f maps the horizontal line $y = c_2$ onto $u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2$ and vertical line onto $\left(u - \frac{1}{2c_1}\right)^2 + v^2 = \left(\frac{1}{2c_1}\right)^2$

<u>Critical Points</u>: If $f'(z)=0$, then the analytic transformation $f(z)$ ceases to be conformal. Such a point is called a critical point of f. So
<u>Critical points are zeros of f'</u>. As f is analytic thus

Critical points are isolated.

<u>Theorem</u> Assume that $f(z)$ is analytic and not constant in a domain D of the complex z-plane. Suppose $f'(z_0) = f''(z_0) = \cdots f^{(n-1)}(z_0) = 0$
while $f^{(n)}(z_0) \neq 0$, $z_0 \in D$.
Then the mapping $z \to f(z)$ magnified n times the angle between two intersecting differentiable arcs that meet at z_0.



As $r \to 0$, $\theta_2-\theta_1 \longrightarrow$ angle between arcs

<u>Proof</u>. Let $z_1(s)$ and $z_2(s)$ be the equations describing arcs intersecting at z_0. If z_1, z_2 are points on these arcs,

$$\left.\begin{array}{l} z_1 - z_0 = re^{i\theta_1} \\ z_2 - z_0 = re^{i\theta_2} \end{array}\right\} \Rightarrow \frac{z_2-z_0}{z_1-z_0} = e^{i(\theta_2-\theta_1)}$$

Angle $\theta_2-\theta_1$ is the angle formed by the linear segments connecting the points z_1-z_0 and z_2-z_0. As $r\to 0$, this angle tends to the angle formed by the two intersecting arcs in the complex z-plane. Similar considerations about angles apply in w-plane.

Hence if θ is angle between arcs in z-plane and ϕ is angle between images of arcs in w-plane

then
$$\theta = \lim_{r\to 0} \arg\left(\frac{z_2-z_0}{z_1-z_0}\right)$$
$$\phi = \lim_{r\to 0} \arg\left(\frac{f(z_2)-f(z_0)}{f(z_1)-f(z_0)}\right)$$

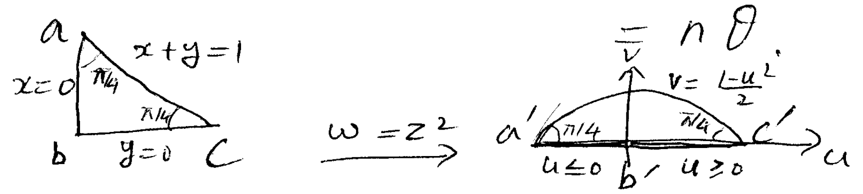
Hence
$$\phi = \lim_{r\to 0} \arg\left\{\left(\frac{\frac{f(z_2)-f(z_0)}{(z_2-z_0)^n}}{\frac{f(z_1)-f(z_0)}{(z_1-z_0)^n}}\right)\left(\frac{z_2-z_0}{z_1-z_0}\right)^n\right\}$$

Using
$$f(z) = f(z_0) + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0)^{n+1} + \cdots$$

we get
$$\lim_{r\to 0} \frac{f(z_2)-f(z_0)}{(z_2-z_0)^n} = \lim_{r\to 0} \frac{f(z_1)-f(z_0)}{(z_1-z_0)^n} = \frac{f^{(n)}(z_0)}{n!}$$

Thus $\phi = \lim_{r\to 0} \arg\left(\frac{z_2-z_0}{z_1-z_0}\right)^n = n \lim_{r\to 0} \arg\left(\frac{z_2-z_0}{z_1-z_0}\right)$
$= n\theta$.

Example.



Let D be the triangular region bounded by $x=0$, $y=0$ and $x+y=1$. The image of D under $w=z^2$ is given by curvilinear triangle shown

$x=0$ is mapped onto $u \le 0$, $v=0$

$y=0$ is mapped onto $u \geq 0, v=0$

$x+y=1$ is mapped onto $v = \frac{1}{2}(1-u^2)$

(We can verify it by taking parametric form $\left.\begin{array}{l} x = 1-t \\ y = t \end{array}\right\}$ of the line and then using $w = z^2$

$$\Rightarrow \quad u(x,y) = (1-t)^2 - t^2 = 1-2t$$
$$v(x,y) = 2\left(-\tfrac{1}{2}\right)(u-1)\left(1+\tfrac{1}{2}(u-1)\right)$$

then Eliminate t)

Now $f(z) = z^2$, $f'(z) = 2z$, $f''(z) = 2$ gives $n=2$ such that $f^{(n)}(z) \neq 0$. Thus angle at critical point $z=0$ should be multiplied by 2. Notice that this is the case as angle at $b' = \pi$ whereas angle at $b = \frac{\pi}{2}$.

<u>Defn</u>. An analytic function $f(z)$ is univalent in a domain D if it takes no value more than once in D.

<u>Remark</u>. A univalent map provides a one to one map of ~~f(~~ D onto $f(D)$. It has a single valued inverse on $f(D)$.

<u>Theorem</u>: Let $f(z)$ be analytic and not constant in a domain D of the complex z-plane. The transformation $w = f(z)$ maps open sets in D onto open sets in $f(D)$.

<u>Theorem</u>. Assume $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$. Then $f(z)$ is <u>univalent</u> in the NHD of z_0

Theorem: (a) Assume that $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$. Then $f(z)$ is univalent in the NHD of z_0. More precisely, f has a unique analytic inverse F in in NHD of $w_0 \equiv f(z_0)$ i.e if z is in the NHD of z_0 then $z = F(w)$ where $w \equiv f(z)$. Similarly if w is sufficiently near w_0, $z \equiv F(w)$ then $w = f(z)$. Moreover $f'(z) F'(w) = 1$, i.e inverse map is also conformal.

(b) Assume that $f(z)$ is analytic at z_0 and that it has a zero of order n i.e $f(z_0) = f'(z_0) = \cdots f^{n-1}(z_0) = 0$ but $f^n(z_0) \neq 0$. Then to each w sufficiently close to $w_0 = f(z_0)$, there corresponds n distinct points z in the in the NHD of z_0, each of which has w as its image under mapping $w = f(z)$. Actually, this mapping can be decomposed in the form

$$w - w_0 = \zeta^n, \quad \zeta = g(z - z_0), \quad g(0) = 0 \text{ where}$$

$g(z)$ is univalent near z_0 and $g(z) = z H(z)$, $H(0) \neq 0$.

Example $w = f(z) = z^2$ from $z \rightarrow w$ plane.

At $w = w_0 = 0$, there are exactly two points (+ve and -ve) corresponding to w_0 close to w_0.

Theorem Let C be a simple closed contour enclosing domain D and let $f(z)$ be analytic on C and in D. Suppose $f(z)$ takes no value more than once on C. Then

(a) $w = f(z)$ takes C enclosing D to a simple closed contour C^* enclosing D^* in w plane. (Conf 10

(b) $w = f(z)$ is one-to-one map from D to D^* and

(c) if z describes C in +ve direction then $w = f(z)$ describes C^* in positive direction.

Physical Applications

We know that the Laplace equation appears in various applications of fluid flow, steady state heat conduction and electrostatic problems. Also real and imaginary parts of an analytic function satisfy the Laplace equation

In order to find unique solution ϕ of Laplace equation $\nabla^2 \phi = 0$, one needs to specify appropriate boundary conditions

Dirichlet Problem: ϕ specified on boundary C

Neumann Problem: normal derivative of ϕ specified on C.

Remark: If solution exists then it must be unique for a Dirichlet problem.

It is possible to obtain solutions of either Dirichlet or Neumann problems using the conformal mappings as follows.

(a) Use a conformal mapping to transform the region R of the z-plane onto a simple region such as a unit circle or a half plane of w plane

(b) Solve the corresponding problem in w plane.

(c) Use this solution and the inverse mapping for to solve the original problem ($f(z)$ is conformal ($f'(z) \neq 0$) thus it has a unique inverse)

Example. Solve Laplace's equation for a function ϕ inside the unit circle that on circumference takes the value

$$\phi_2 \,, \quad 0 \le \theta < \pi$$
$$\phi_1 \,, \quad \pi \le \theta < 2\pi.$$

(Find steady state heat distribution inside a disk with a prescribed temperature ϕ on the boundary)

We shall use a special case of the so-called bi-linear transformations $f(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$.

We take $$w = \frac{i(1-z)}{(1+z)}$$

This gives $$u = \frac{2y}{(1+x)^2+y^2}, \quad v = \frac{1-(x^2+y^2)}{(1+x)^2+y^2}$$

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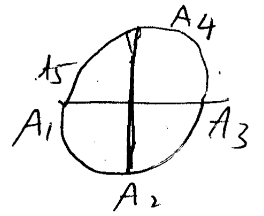
z on the unit circle has the form

$z = e^{i\theta}$ then $w(z) = u = \dfrac{\sin\theta}{1+\cos\theta}$

$(v=0)$.



$\overset{\frown}{A_1A_2A_3}$ is mapped onto negative real axis. $\overset{\frown}{A_3A_4A_5}$ is mapped onto +ve real axis.

Solution in w-plane. Let $w = \rho e^{i\psi}$.

The function $a\psi + b$, a b are real constants the real part of the analytic function $-ai\log w + b$ in the upper half plane.

$$(-ai(\log|\rho e^{i\psi}| + i\psi)) + b$$

$$\text{Real part} = a\psi + b$$

Thus $a\psi+b$, being real part of an analytic function is harmonic and hence a solution of the Laplace equation in the upper w plane.

~~Also when u<0~~

To satisfy boundary conditions $\begin{cases} \phi = \phi_1 & \text{for } u<0, v=0 \text{ (i.e. } \psi=\pi) \\ \phi = \phi_2, & u>0, v=0) \text{ i.e. } \psi=0 \end{cases}$

we have

$$\phi = \phi_2 - (\phi_2 - \phi_1)\frac{\psi}{\pi}, \text{ But } \psi = \arg w \text{ so}$$

$$\phi = \phi_2 - \frac{\phi_2 - \phi_1}{\pi}\tan^{-1}\left(\frac{v}{u}\right).$$

Due to uniqueness property, this solution is unique.

$$\phi(x,y) = \phi_2 - \frac{\phi_2-\phi_1}{\pi}\tan^{-1}\left[\frac{1-(x^2+y^2)}{2y}\right]$$