Remarks ① If $\nu = 0$,

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! n!}$$

For $\nu = 1$,

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n! (n+1)!}$$

We note that $J_0(0) = 1$, $J_1(0) = 0$. $J_0(x)$ and $J_1(x)$ behave rather like $\cos x$ and $\sin x$ respectively except that they have bigger denominator. For this reason, they are sometimes called damped cosine and damped sine functions respectively. One can also notice $J_0(-x) = J_0(x)$ and $J_1(-x) = -J_1(x)$.

② If $\nu = \pm m$, the $J_m(x)$ and J_{-m} are linearly dependent. In fact,

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-m}}{2^{2n-m} n! (n-m)!}$$

The first non zero term will occur when $n = m$ (so that $(n-m)! = 0! = 1$). Thus

$$J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n x^{2n-m}}{2^{2n-m} n! (n-m)!}$$

Change index inside Σ to $n \to n+m$, so that $2n-m \to 2(n+m) - m = 2n+m$, $n-m \to n$ and variation of Σ becomes $n = 0$ to ∞,

$$J_{-m} = \sum_{n=0}^{\infty} \frac{(-1)^{n+m} x^{2n+m}}{2^{2n+m} (n+m)! n!} = (-1)^m J_m(x).$$

(3) For ν a rational fraction, $J_\nu(x)$ and $J_{-\nu}(x)$ are two linearly independent solutions of the Bessel equation of order ν, so that

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$$ is the general solution in this case.

__Examples__

(i) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\,\sin x$

(2) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\,\cos x$

(3) $J_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[-\sin x - \dfrac{\cos x}{x}\right]$

__Solution:__

(1) $$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1/2}}{2^{2n+1/2}\, n!\, \Gamma(n+\frac{1}{2}+1)} = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}\, n!\, \Gamma(n+\frac{1}{2}+1)} \quad \text{—(i)}$$

Now $\Gamma(n+\frac{1}{2}+1) = (n+\frac{1}{2})\Gamma(n+\frac{1}{2}) = (n+\frac{1}{2})\Gamma(n-\frac{1}{2}+1)$

$$= (n+\tfrac{1}{2})(n-\tfrac{1}{2})\Gamma(n-\tfrac{1}{2}) = (n+\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \left(\frac{2n+1}{2}\right)\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots \Gamma(\tfrac{1}{2})$$

$$= \frac{(2n+1)(2n-1)(2n-3)\cdots 3\cdot 1}{2^{n+1}}\,\Gamma(\tfrac{1}{2}) \quad \text{—(2)}$$

Also $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$$= \frac{2n}{2}\left(\frac{2n-2}{2}\right)\left(\frac{2n-4}{2}\right)\cdots\left(\frac{6}{2}\right)\cdot\left(\frac{4}{2}\right)\cdot\left(\frac{2}{2}\right)$$

$$= \frac{2n(2n-2)(2n-4)\cdots 6\cdot 4\cdot 2}{2^n} \quad \text{—(3)}$$

From (2) & (3) $$2^{2n+1}\left\{n!\,\Gamma(n+\tfrac{1}{2}+1)\right\} = \frac{(2n+1)\,2n\cdots 4\cdot 3\cdot 2\cdot 1}{\cancel{2^n}\cdot \cancel{2^{n+1}}}\,\Gamma(\tfrac{1}{2}) = (2n+1)!\,\Gamma(\tfrac{1}{2})$$

From (1) $$J_{1/2}(x) = \sqrt{\frac{2}{x}}\,\frac{1}{\Gamma(1/2)}\,\sin x = R.H.S \text{ as } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Derivative Formulae

① $\left[x^{-\nu} J_\nu(x)\right]' = -x^{-\nu} J_{\nu+1}(x)$

② $\left[x^{\nu} J_\nu(x)\right]' = x^{\nu} J_{\nu-1}(x)$

Proof ① $x^{-\nu} J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+\nu} n! \, \Gamma(n+\nu+1)}$

Note that $n=0$ term is a constant. Differentiation gives

$$\left[x^{-\nu} J_\nu(x)\right]' = \sum_{n=1} \frac{(-1)^n 2n \; x^{2n-1}}{2^{2n+\nu} n! \; \Gamma(n+\nu+1)}$$

Changing index

$$\left[x^{-\nu} J_\nu(x)\right]' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(n+1) \, x^{2n+1}}{2^{2n+\nu+2} (n+1)! \; \Gamma(n+\nu+2)}$$

$$\left[x^{-\nu} J_\nu(x)\right]' = -x^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+\nu+1}}{2^{2n+\nu+1} n! \; \Gamma(n+\overline{\nu+1}+1)}$$

$$= -x^{-\nu} J_{\nu+1}(x).$$

② $x^{\nu} J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2\nu}}{2^{2n+\nu} n! \, \Gamma(n+\nu+1)}$

In this case $n=0$ term is not a constant!

$$\left[x^{\nu} J_\nu(x)\right]' = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2\nu) \, x^{2n+2\nu-1}}{2^{2n+\nu} n! \; \Gamma(n+\nu+1)}$$

$$= x^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n \, 2(n+\nu) \, x^{2n+\nu-1}}{2^{2n+\nu} n! \; \frac{\Gamma(n+\nu)}{n+\nu}} = x^{\nu} J_{\nu-1}(x).$$

Special Cases

From ①, $\nu = 0$ gives $J_0'(x) = -J_1(x)$

From ② $\nu = 1$ gives $[x J_1(x)]' = x J_0(x)$

Solution of Example (2) Page 66 From ex ①

$$x^{1/2} J_{1/2}(x) = \sqrt{\frac{2}{\pi}} \sin x$$

Now $[x^{1/2} J_{1/2}(x)]' = x^{1/2} J_{1/2 - 1}(x) = x^{1/2} J_{-1/2}(x)$

$$= \sqrt{\frac{2}{\pi}} \cos x$$

Hence $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

Recurrence Relations:

Consider ① P 67

$$[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x).$$

Carrying out derivative operation on L.H.S

$$x^{-\nu} J_\nu'(x) - \nu x^{-\nu-1} J_\nu(x) = -x^{-\nu} J_{\nu+1}(x)$$

Multiplying by x^ν

$$J_\nu'(x) - \nu x^{-1} J_\nu(x) = -J_{\nu+1}(x) \quad \text{—— (3)}$$

Similarly from ② P 67

$$x^\nu J_\nu'(x) + \nu x^{\nu-1} J_\nu(x) = x^\nu J_{\nu-1}(x)$$

or $J_\nu'(x) + \nu x^{-1} J_\nu(x) = J_{\nu-1}(x)$ —— ④

Adding ③ and ④

$$\boxed{2 J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)}$$

Subtracting ③ from ④

$$2\nu x^{-1} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$$

or

$$\boxed{2\nu J_\nu(x) = x\left(J_{\nu-1}(x) + J_{\nu+1}(x)\right)}$$

Solution of Example ③ Page 66

In the above formula put $\nu = -\frac{1}{2}$

$$\cancel{2}\left(-\frac{1}{\cancel{2}}\right) J_{-1/2}(x) = x\, J_{-3/2} + x J_{1/2}(x)$$

or $x J_{-3/2} = -x J_{1/2}(x) - J_{-1/2}(x)$

Using results in ex(1) and (2) p 66,

$$x J_{-3/2}(x) = x\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

or $J_{-3/2} x = \sqrt{\frac{2}{\pi x}}\left[-\sin x - \frac{\cos x}{x}\right]$.

Integral Formulae

(a) $\int x^{-\nu} J_{\nu+1}(x)\, dx = -x^{-\nu} J_\nu(x) + C$

(b) $\int x^{\nu} J_{\nu-1}(x)\, dx = x^{\nu} J_\nu(x) + C$

In particular

$$\int x J_0(x)\, dx = x J_1(x) + C$$

$$\int J_1(x)\, dx = -J_0(x) + C$$

Bessel Equation with Eigenvalues

Consider $x^2 y'' + x y' + (\lambda x^2 - \nu^2) y = 0$ — (1)

In Sturm - Liouville form

$$(x y')' + \left(\frac{-\nu^2}{x} + \lambda x\right) y = 0 \quad , \quad 0 < x < l$$

No boundary condition at $x = 0$. At $x = l$, we may have a boundary condition such as $y(l) = 0$.

Note also $p(x) = x$.

Put $t = \sqrt{\lambda}\, x$, $\frac{dt}{dx} = \sqrt{\lambda}$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \sqrt{\lambda}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\sqrt{\lambda}\frac{dy}{dt}\right)\frac{dt}{dx} = \lambda \frac{d^2y}{dt^2}$$

D.E (1) becomes

$$\frac{t^2}{\lambda}\cdot\lambda\frac{d^2y}{dt^2} + \frac{t}{\sqrt{\lambda}}\sqrt{\lambda}\frac{dy}{dt} + \left(\frac{\lambda t^2}{\lambda} - \nu^2\right) y(t) = 0$$

or $$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)\, y(t) = 0$$

which is the Bessel equation of order ν, The solution being $J_\nu(\cancel{t}) = J_\nu(\sqrt{\lambda}\, x)$ (second solution $J_{-\nu}(\sqrt{\lambda} x)$ if ν is not an integer). Otherwise, the second solution (by Frobenius method) can be found. We put it in a particular form and denote it by $Y_n(\sqrt{\lambda} x)$ — Bessel Function of second kind. However, as it contains a $\ln$ term,

$y(x) = C_1 J_n(\sqrt{\lambda}x) + C_2 J_{-n}(\sqrt{\lambda}\,x)$

gives $C_2 = 0$ (so that solution is bounded at $x=0$.)

If $n=0$, we get

$y = C_1 J_0 \sqrt{\lambda}\,x$. If $y(l)=0$, this

gives $C_1 J_0(\sqrt{\lambda}\,l) = 0$

$C_1 \neq 0$, $J_0(\sqrt{\lambda}\,l) = 0$.

If x_K, $K=1,2,3\ldots$ are zeros of $J_0(x)$ then

$\sqrt{\lambda}\,l = x_K$ or $\sqrt{\lambda_k} = \dfrac{x_k}{l}$, (label K being used)

The corresponding eigenfunctions

$$C_K\, J_0(\sqrt{\lambda_K}\, x) \quad , \quad K=1,2,3,\ldots$$

Orthogonality Relation

$$\int_0^l x\, J_0(\sqrt{\lambda_K}\, x)\, J_0(\sqrt{\lambda_m}\, x)\, dx = 0$$

$$K \neq m.$$

General Equation :

$$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2-\nu^2c^2}{x^2}\right) y = 0$$

has one solution $x^a J_\nu(b x^c)$

Example. $$y'' + \frac{3}{x} y' + \left(16x^2 - \frac{5}{4x^2}\right) y = 0$$

Comparing with the general equation

$$-(2a-1) = 3 \quad \Rightarrow \quad -2a+1 = 3 \text{ or } -2a = 2 \Rightarrow a = -1$$

$$b^2c^2 = 16 \qquad (*)$$

$$2c-2 = 2 \quad \Rightarrow \quad c = 2$$

$$a^2 - \nu^2c^2 = -\frac{5}{4} \qquad (**)$$

$$* \Rightarrow \quad b^2(4) = 16 \quad \Rightarrow \quad b^2 = 4 \Rightarrow b = 2$$

$$** \Rightarrow \quad 1 - 4\nu^2 = -\frac{5}{4} \quad \Rightarrow \quad -4\nu^2 = -\frac{5}{4} - 1 = \frac{-5-4}{4} = -\frac{9}{4}$$

$$\Rightarrow \nu^2 = \frac{9}{16} \quad \text{or } \nu = \frac{3}{4}$$

Hence $x^{-1} J_{3/4}(2x^2)$ is one solution.

General solution

$$y = C_1 x^{-1} J_{3/4}(2x^2) + C_2 x^{-1} J_{-3/4}(2x^2).$$

10. $\dfrac{\partial u}{\partial t} = 9\dfrac{\partial^2 u}{\partial x^2} \quad (x > 0, t > 0)$

$u(x, 0) = 0 \quad (x > 0)$

$u(0, t) = f(t) \quad (t > 0)$

11. $\nabla^2 u = 0 \quad (x > 0, 0 < y < 1)$

$u(0, y) = y^2(1 - y) \quad (0 < y < 1)$

$u(x, 0) = u(x, 1) = 0 \quad (x > 0)$

12. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - u \quad (x > 0, t > 0)$

$\dfrac{\partial u}{\partial x}(0, t) = f(t) \quad (t > 0)$

$u(x, 0) = 0 \quad (x > 0)$

13. $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + tu = 0 \quad (x > 0, t > 0)$

$u(x, 0) = xe^{-x} \quad (x > 0)$

$u(0, t) = 0 \quad (t > 0)$

14. $\nabla^2 u = 0 \quad (-\infty < x < \infty, 0 < y < 1),$

$u(x, 0) = 0$ for $x < 0$ and $u(x, 0) = e^{-ax}$ for $x > 0$

$u(x, 1) = 0 \quad (-\infty < x < \infty)$

Here a is a positive constant.

15. $\nabla^2 u = 0 \quad (-\infty < x < \infty, 0 < y < 1)$

$\dfrac{\partial u}{\partial y}(x, 0) = 0 \quad (-\infty < x < \infty)$

$u(x, 1) = e^{-x^2} \quad (-\infty < x < \infty)$

16.8 The Heat Equation in an Infinite Cylinder

This section marks the beginning of four sections in which the solution of a boundary value problem requires one or more special techniques such as a change of variables, a multiple series or a special function. We will also make use of the Sturm-Liouville theorem (Section 6.1) and the idea of eigenfunction expansions (Section 6.2).

Suppose we want the temperature distribution in a solid, infinitely long, homogeneous circular cylinder of radius R. Let the z-axis be along the axis of the cylinder (Figure 16.55). In cylindrical coordinates the heat equation is

$$\frac{\partial u}{\partial t} = a^2\nabla^2 u = a^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right].$$

This is a formidable equation to engage at this point, so we will assume that the temperature at any point in the cylinder depends only on the time t and the distance r from the z-axis, the axis of the cylinder. This means that $\partial u/\partial\theta = \partial u/\partial z = 0$ and the heat equation is

$$\frac{\partial u}{\partial t} = a^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right].$$

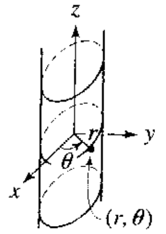

Figure 16.55

Case 3 $\lambda > 0$.

Write $\lambda = k^2$, with $k > 0$. Then $T' + a^2k^2T = 0$ with general solution

$$T(t) = ce^{-a^2k^2t}.$$

The equation for F is

$$F'' + \frac{1}{r}F' + k^2F = 0,$$

which we can write as

$$r^2F'' + rF'(r) + k^2r^2F = 0.$$

From the material on Bessel functions in Section 6.6, the general solution is

$$F(r) = AJ_0(kr) + BY_0(kr).$$

Since $Y_0(kr) \to -\infty$ as $r \to 0$ we must choose $B = 0$, so

$$F(r) = AJ_0(kr).$$

For each $k > 0$, we now have a function

$$u_k(r, t) = a_kJ_0(kr)e^{-a^2k^2t},$$

which satisfies the heat equation in cylindrical coordinates.

For $t > 0$, we need

$$u_k(R, t) = a_kJ_0(kR)e^{-a^2k^2t} = 0.$$

To avoid a trivial solution, we must have a_k nonzero. The only way this equation can be satisfied with this constraint is to choose k so that

$$J_0(kR) = 0.$$

Now, the graph of $J_0(x)$ crosses the x-axis at infinitely many positive values of x (see Figure 6.12). Let these values be $z_1, z_2, \ldots$, written in increasing order for convenience. We should choose values of k so that

$$kR = z_n$$

or

$$k = \frac{z_n}{R}.$$

For $n = 1, 2, \ldots$, this gives the admissible values of k. Now we have

$$u_n(r, t) = a_nJ_0\left(\frac{z_nr}{R}\right)e^{-a^2z_n^2t/R^2}.$$

Each $u_n(r, t)$ satisfies the heat equation and the condition that $u_n(R, t) = 0$ for all times. We must satisfy the condition $u(r, 0) = f(r)$. In general, it is not possible to do this with a finite linear combination of the u_n's, and we must usually attempt an infinite superposition

$$u(r, t) = \sum_{n=1}^{\infty} u_n(r, t) = \sum_{n=1}^{\infty} a_nJ_0\left(\frac{z_nr}{R}\right)e^{-a^2z_n^2t/R^2}.$$

We now need to choose the a_n's so that

$$u(r, 0) = f(r) = \sum_{n=1}^{\infty} a_nJ_0\left(\frac{z_nr}{R}\right). \tag{16.51}$$

This is an expansion of $f(r)$ in a series of Bessel functions, which are the eigenfunctions of the Sturm-Liouville problem for $F(r)$. As with Fourier series of sines and cosines, the key to choosing the coefficients is the orthogonality of these eigenfunctions with respect to the weight function r:

$$\int_0^R r J_0\left(\frac{z_n r}{R}\right) J_0\left(\frac{z_m r}{R}\right) dr = 0 \quad \text{if} \quad n \neq m. \tag{16.52}$$

This relationship follows from the Sturm-Liouville theorem (Theorem 6.1 of Section 6.1 or note page 260), or from Lommel's integrals (Problem 24 of Section 6.5) upon letting $x = r, \alpha = z_n/R$, $\beta = z_m/R$, and inserting the limits of integration.

Although the formula for the a_n's is actually given by equation (6.11) in Section 6.2, we will derive it independently here by an informal argument like that used for the coefficients in Fourier trigonometric series in Section 14.1. Multiply equation (16.51) by $r J_0(z_m r/R)$ and integrate the resulting equation from 0 to R, interchanging the integration and the summation to get

$$\int_0^R r f(r) J_0\left(\frac{z_m r}{R}\right) dr = \sum_{n=1}^{\infty} a_n \int_0^R r J_0\left(\frac{z_n r}{R}\right) J_0\left(\frac{z_m r}{R}\right) dr.$$

Because of the orthogonality (equation (16.52)) all terms on the right are zero, except possibly the term in which $n = m$, yielding

$$\int_0^R r f(r) J_0\left(\frac{z_m r}{R}\right) dr = a_m \int_0^R r \left[J_0\left(\frac{z_m r}{R}\right)\right]^2 dr;$$

hence,

$$a_m = \frac{\int_0^R r f(r) J_0(z_m r/R) dr}{\int_0^R r\,[J_0(z_m r/R)]^2\, dr}.$$

With this choice of the coefficients, the series of equation (16.51) converges to $f(r)$, assuming that f satisfies certain reasonable conditions (see the convergence theorem of Section 6.2). The solution of the boundary value problem is

$$u(r,t) = \sum_{n=1}^{\infty} \left[\frac{\int_0^R \xi f(\xi) J_0(z_n \xi/R)\, d\xi}{\int_0^R r\,[J_0(z_n \xi/R)]^2\, d\xi}\right] J_0\left(\frac{z_n r}{R}\right) e^{-a^2 z_n^2 t/R^2}.$$

In the next section we will apply Fourier-Legendre series to the solution of a boundary value problem modeling heat conduction in a solid sphere.

roblems

A homogeneous, circular cylinder of radius 2 and semi-infinite length has its base, which is sitting on the plane $z = 0$, maintained at a constant positive temperature K. The lateral surface is kept at temperature zero. Determine the steady-state temperature of the cylinder if it has a thermal diffusivity of a^2, assuming that the temperature at any point depends only on the height z above the base and the distance r from the axis of the cylinder. *Hint:* Let the axis of the cylinder be the z-axis. Set up the boundary value problem and separate the variables, then use the zero boundary condition.

2. Redo Problem 1 with the assumption that the lateral surface is maintained at constant positive temperature $L < K$.
3. A solid cylinder is bounded by $z = 0$ and $z = L$ and by $r = R$ (cylindrical coordinates). The cylinder's lateral surface is insulated, while the top is kept at temperature 2 and the bottom at temperature zero. Find the steady-state temperature distribution in the cylinder.
4. Determine the temperature distribution in a circular cylinder of radius R with insulated top and bottom under the assumption that the temperature