

Example From the above formula (5) we can get $P_3(x)$ by putting $n = 3$. In this case $m = \left[\frac{3}{2}\right] = 1$. (5)

$$P_3(x) = \frac{6!}{2^3 3!3!} x^3 - \frac{4!}{2^3 2!} x = \frac{5}{2} x^3 - \frac{3}{2} x.$$

Rodrigue Formula: This formula provides another way of generating Legendre Polynomials.

Step 1

$$(x^2-1)^n = x^{2n} - n x^{2n-1} - \frac{(-1)^k n!}{(k-1)!(n-k)!} x^{2n-k+1} + \dots$$

$$= \sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!} x^{2n-2k}$$

Step 2: $\frac{d^n}{dx^n} [x^{2n-2k}] = \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$

Step 3: From above

$$\frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!} x^{2n-2k} = P_n(x).$$

Hence $P_n(x) = \frac{1}{2^n n!} \left\{ \frac{d^n}{dx^n} (x^2-1)^n \right\}$

This is ~~the~~ Rodrigue's formula.

Example: $n=1$ gives

$$P_1(x) = \frac{1}{2 \cdot 1!} \left[\frac{d}{dx} (x^2-1) \right] = \frac{1}{2} \cdot 2x = x$$

$$n=2,$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} [(x^2-1)^2]$$

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$$= \frac{1}{8} \frac{d}{dx} [2(x^2-1) \cdot 2x]$$

$$= \frac{1}{8} \frac{d}{dx} [4x^3 - 4x] = \frac{1}{8} [12x^2 - 4]$$

$$= \frac{3}{2} x^2 - \frac{1}{2}.$$

Similarly $n=3$ can be put to obtain $P_3(x)$.

Generating Function: Consider $P(x,t) = (1-2xt+t^2)^{-1/2}$

Using the Binomial formula, $(1-z)^{-1/2} = 1 + \frac{1}{2}z + \frac{1}{2!} \frac{1 \cdot 3}{2^2} z^2 + \frac{1}{3!} \frac{1 \cdot 3 \cdot 5}{2^3} z^3 + \dots$

For $z = 2xt - t^2$, we can write

$$P(x,t) = (1-2xt+t^2)^{-1/2} = 1 + \frac{1}{2}(2xt - t^2)$$

$$+ \frac{3}{8}(2xt - t^2)^2 + \frac{5}{16}(2xt - t^2)^3 + \dots$$

Rearrange the terms to write

$$P(x,t) = 1 + xt + \left(-\frac{1}{2} + \frac{3}{2}x^2\right)t^2 + \left(-\frac{3}{2}x + \frac{5}{2}x^3\right)t^3$$

+ ---

$$= P_0(x) + P_1(x)t + P_2(x)t^2 + \dots$$

Thus $P(x,t)$ is the generating function for $P_n(x)$.

Recurrence Relation

(53)

The recurrence relation for $P_n(x)$ is
$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$
$$n > 0.$$

To show this result, consider

$$P(x,t) = (1-2xt+t^2)^{-1/2} \quad \text{--- ①}$$

$$\frac{\partial P}{\partial t} = (x-t)(1-2xt+t^2)^{-3/2}$$

or multiplying by $(1-2xt+t^2)$ and using ①
$$(1-2xt+t^2)\frac{\partial P}{\partial t} - (x-t)P = 0$$

Putting $P(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$, we get

$$\sum_{n=1}^{\infty} (1-2xt+t^2)nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} (x-t)P_n(x)t^n = 0$$

$$n \sum_{n=1}^{\infty} nP_n(x)t^{n-1} + \sum_{n=1}^{\infty} -2xnP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

$$- \sum_{n=0}^{\infty} xP_n(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^{n+1} = 0$$

Change index in $\sum_{n=0}^{\infty}$ so as t^n is the power

of t in all of them

$$\sum_{n=0}^{\infty} (n+1)P_{n+1}(x)t^n + \sum_{n=1}^{\infty} -2xnP_n(x)t^n + \sum_{n=2}^{\infty} (n-1)P_{n-1}(x)t^n$$

$$- \sum_{n=0}^{\infty} xP_n(x)t^n + \sum_{n=1}^{\infty} P_{n-1}(x)t^n = 0$$

Combining as on Σ while taking extra terms out

$$\begin{aligned}
 & P_1(x) + 2 P_2(x)t - 2x P_1(x)t \\
 & - x P_0(x) - x P_1(x)t + P_0(x) \\
 & + \sum_{n=2}^{\infty} \{ (n+1) P_{n+1} - (2n+1)x P_n + n P_{n-1} \} t^n = 0
 \end{aligned}$$

$$\text{Coeff}(t^n) = 0 \implies$$

$$(n+1) P_{n+1}(x) - (2n+1)x P_n + n P_{n-1}(x) = 0$$

which is the required result.

Example. Using $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ we get obtain $P_3(x)$ from this result by putting $n = 2$.

$$3 P_3(x) - 5x P_2 + 2 P_1 = 0$$

$$3 P_3(x) = 5x \left(\frac{3}{2}x^2 - \frac{1}{2} \right) - 2x$$

$$= \frac{15}{2}x^3 - \frac{5}{2}x - 2x = \frac{15}{2}x^3 - \frac{9}{2}x$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

Norm of $P_n(x)$: As we have seen that

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$\text{where } a_n = \frac{\int_{-1}^1 f(x) P_n(x) dx}{\int_{-1}^1 P_n^2(x) dx} = \frac{\int_{-1}^1 f(x) P_n(x) dx}{\|P_n(x)\|^2}$$

It is therefore useful to have a formula (55)
for $\|P_n(x)\|^2$.

Consider the recurrence relation

$$(n+1)P_{n+1} + nP_{n-1} = (2n+1)xP_n(x) \quad \text{--- (a)}$$

$n \rightarrow n-1$ gives

$$nP_n + (n-1)P_{n-2} = (2n-1)xP_{n-1} \quad \text{--- (b)}$$

Take scalar product of (a) with P_{n-1} and
of (b) with P_n

$$(n+1)\langle P_{n+1}, P_{n-1} \rangle + n\langle P_{n-1}, P_{n-1} \rangle = (2n+1)\langle xP_n, P_{n-1} \rangle$$

$$n\langle P_n, P_n \rangle + (n-1)\langle P_{n-2}, P_n \rangle = (2n-1)\langle xP_{n-1}, P_n \rangle$$

Using orthogonality

$$\langle P_{n+1}, P_{n-1} \rangle = 0$$
$$\langle P_{n-2}, P_n \rangle = 0,$$

so

$$n\langle P_{n-1}, P_{n-1} \rangle = (2n+1)\langle xP_n, P_{n-1} \rangle$$

and $n\langle P_n, P_n \rangle = (2n-1)\langle xP_{n-1}, P_n \rangle$

Note that $\langle xP_n, P_{n-1} \rangle$ and $\langle xP_{n-1}, P_n \rangle$

both are equal to $\int_{-1}^1 xP_n(x)P_{n-1}(x)dx$

Eliminating this we get

$$(2n+1)\|P_n\|^2 = (2n-1)\|P_{n-1}\|^2$$

From this relation we can write

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$$3 \|P_1\|^2 = 1 \|P_0\|^2$$

$$5 \|P_2\|^2 = 3 \|P_1\|^2$$

$$(2n-1) \|P_{n-1}\|^2 = (2n-3) \|P_{n-2}\|^2$$

$$(2n+1) \|P_n\|^2 = (2n-1) \|P_{n-1}\|^2$$

Taking product and cancelling repeated terms on both sides

$$(2n+1) \|P_n\|^2 = \|P_0\|^2$$

$$\text{or } \|P_n\|^2 = \frac{\|P_0\|^2}{(2n+1)}$$

$$\text{Now } \|P_0\|^2 = \int_{-1}^1 1 \cdot dx = [x]_{-1}^1 = 2$$

$$\text{Hence } \|P_n\|^2 = \frac{2}{2n+1}$$

$$\text{or } \|P_n\| = \sqrt{\frac{2}{2n+1}}$$

Exercise Express (a) $f(x) = \begin{cases} 1 & , -1 < x < 0 \\ 0 & , 0 < x < 1 \end{cases}$

as Legendre polynomials (b) $f(x) = 1$

Solution of Boundary value Problem

(5)

Let us consider steady state temperature distribution in a sphere of radius a . If radial symmetry is assumed, the temperature u is independent of ϕ and the Laplace equation in terms of spherical coordinates is

$$r \frac{\partial^2 (ru)}{\partial r^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u}{\partial \theta}) = 0, \quad r < a \quad \text{--- (1)}$$
$$0 < \theta < \pi$$

with temperature at surface given by

$$u(a, \theta) = F(\theta), \quad 0 < \theta < \pi. \quad \text{--- (2)}$$

Assume $u(r, \theta) = R(r) G(\theta)$, we obtain from (1)

$$\left[r \frac{\partial^2 (r R(r))}{\partial r^2} \right] G(\theta) + \frac{R(r)}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta G'(\theta)) = 0$$

$$\frac{r \frac{d^2 (r R(r))}{dr^2}}{R(r)} = - \frac{1}{\sin \theta} \frac{d(\sin \theta G'(\theta))}{d\theta} = \lambda$$

where λ is the separation constant.

This gives

$$r \frac{d^2 (r R(r))}{dr^2} - \lambda R = 0 \quad \text{--- (3)}$$

$$\text{and} \quad \frac{d}{d\theta} (\sin \theta \frac{dG}{d\theta}) + \lambda \sin \theta G(\theta) = 0 \quad \text{--- (4)}$$

In (4), we put $x = \cos \theta$, $\frac{dx}{d\theta} = -\sin \theta$

so that $\frac{dG}{d\theta} = \frac{dG}{dx} \frac{dx}{d\theta} = -\frac{dG}{dx} \sin \theta$.

hence $\sin \theta \frac{dG}{d\theta} = -\sin^2 \theta \frac{dG}{dx} = -(1-\cos^2 \theta) \frac{dG}{dx} = -(1-x^2) \frac{dG}{dx}$

Also we notice $\frac{d}{d\theta} = -\sin \theta \frac{d}{dx}$

Thus $\frac{d}{d\theta} (\sin \theta \frac{dG}{d\theta}) = \sin \theta \frac{d}{dx} [-(1-x^2) \frac{dG}{dx}]$

Hence (4) gives

$\sin \theta \frac{d}{dx} [(1-x^2) \frac{dG}{dx}] + \lambda \sin \theta G(x) = 0$ (5)

Equation (5) can be recognized as the Legendre equation. Thus $\lambda_n = n(n+1)$, $n=0,1,2, \dots$

$G(x) = P_n(x)$ " "

or in terms of θ , $G_n(\theta) = P_n(\cos \theta)$

Equation (3) for $\lambda = n(n+1)$ gives

$r \frac{d^2}{dr^2} (r R(r)) - n(n+1) R = 0$ or

$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - n(n+1) R = 0$

Using $t = \ln r$, $\frac{dt}{dr} = \frac{1}{r}$

$\frac{dR}{dr} = \frac{dR}{dt} \frac{dt}{dr} = \frac{1}{r} \frac{dR}{dt}$

$\frac{d^2 R}{dr^2} = \frac{1}{r^2} \frac{d^2 R}{dt^2} - \frac{1}{r^2} \frac{dR}{dt}$

This gives

$$\frac{d^2 R}{dt^2} + \frac{dR}{dt} - n(n+1)R = 0$$

It has solution

$$R(r) = C_1 r^n + C_2 r^{-n-1}$$

To keep this bounded at $r=0$, we must have

$C_2 = 0$. i.e. $R_n(r) = C_n r^n$. Using $u_n(r, \theta)$ and

By principle of superposition,

$$u(r, \theta) = \sum_{n=0}^{\infty} B_n r^n P_n(\cos \theta)$$

Using $u(a, \theta) = F(\theta)$, we get

$$F(\theta) = \sum_{n=0}^{\infty} B_n a^n (P_n \cos \theta)$$

The constant $B_n a^n$ can be found to be

$$B_n a^n = \frac{2n+1}{2} \int_0^{\pi} F(\theta) P_n(\cos \theta) \sin \theta d\theta.$$