<u>Example</u> From the above formula (5) we can get $P_3(x)$ by putting $n = 3$. In this case $m = \left[\frac{3}{2}\right] = 1$.

$$P_3(x) = \frac{6!}{2^3\, 3!\, 3!} x^3 - \frac{4!\, x}{2^3\, 2!} = \frac{5}{2} x^3 - \frac{3}{2} x.$$

<u>Rodrigue Formula</u>: This formula provides another way of generating Legendre Polynomials.

<u>Step 1</u>

$$(x^2-1)^n = x^{2n} - n\, x^{2n-1} - \frac{(-1)^{k-1}\, n!}{(k-1)!\,(n-k+1)!} x^{2n-k+1} + \cdots$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n!}{k!\,(n-k)!} x^{2n-2k}$$

<u>Step 2</u>:

$$\frac{d^n}{dx^n}\left[x^{2n-2k}\right] = \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

<u>Step 3</u>: From above

$$\frac{1}{2^n\, n!} \frac{d^n}{dx^n} \sum_{k=0}^{m} (-1)^k \frac{n!}{k!\,(n-k)!} x^{2n-2k} = P_n(x).$$

Hence $P_n(x) = \frac{1}{2^n\, n!} \left\{ \frac{d^n}{dx^n} (x^2-1)^n \right\}$

This is ~~Rog~~ Rodrigues formula.

<u>Example</u>: $n = 1$ gives

$$P_1(x) = \frac{1}{2\, 1!} \left[\frac{d}{dx}(x^2-1) \right] = \frac{1}{2} \cdot 2x = x$$

$n = 2$,

$$P_2(x) = \frac{1}{2^2\,2!}\,\frac{d^2}{dx^n}\left[(x^2-1)^2\right]$$

$$= \frac{1}{8}\,\frac{d}{dx}\left[2(x^2-1)\cdot 2x\right]$$

$$= \frac{1}{8}\,\frac{d}{dx}\left[4x^3 - 4x\right] = \frac{1}{8}\left[12x^2 - 4\right]$$

$$= \frac{3}{2}x^2 - \frac{1}{2}.$$

Similarly $n = 3$ can be put to obtain $P_3(x)$.

<u>Generating Function</u>: Consider $P(x,t) = (1-2xt+t^2)^{-1/2}$

Using the Binomial formula, $(1-z)^{-1/2} = 1 + \frac{1}{2}z + \frac{1}{2!}\frac{1\cdot 3}{2^2}z^2 + \frac{1}{3!}\frac{1\cdot 3\cdot 5}{2^3}z^3 + \cdots$

For $z = 2xt - t^2$, we can write

$$P(x,t) = (1-2xt+t^2)^{-1/2} = 1 + \frac{1}{2}(2xt - t^2) + \frac{3}{8}(2xt - t^2)^2 + \frac{5}{16}\cdot(2xt - t^2)^3 + \cdots$$

Rearrange the terms to write

$$P(x,t) = 1 + xt + \left(-\frac{1}{2} + \frac{3}{2}x^2\right)t^2 + \left(-\frac{3}{2}x + \frac{5}{2}x^3\right)t^3 + \cdots$$

$$= P_0(x) + P_1(x)\,t + P_2(x)\,t^2 + \cdots$$

Thus $P(x,t)$ is the generating function for $P_n(x)$.

Recurrence Relation

The recurrence relation for $P_n(x)$ is

$$(n+1)P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0$$

$$n > 0.$$

To show this result, consider

$$P(x,t) = (1-2xt+t^2)^{-1/2} \quad \text{—— (1)}$$

$$\frac{\partial P}{\partial t} = (x-t)\,(1-2xt+t^2)^{-3/2}$$

or multiplying by $(1-2xt+t^2)$ and using (1)

$$(1-2xt+t^2)\frac{\partial P}{\partial t} - (x-t)\,P = 0$$

Putting $P(x,t) = \sum_{n=0}^{\infty} P_n(x)\,t^n$, we get

$$\sum_{n=1}^{\infty}(1-2xt+t^2)\,n\,P_n(x)\,t^{n-1} - \sum_{n=0}(x-t)\,P_n(x)\,t^n = 0$$

or $$\sum_{n=1}^{\infty} n\,P_n(x)\,t^{n-1} + \sum_{n=1}^{\infty} -2xn\,P_n(x)\,t^n + \sum_{n=1} n\,P_n(x)\,t^{n+1}$$

$$- \sum_{n=0}^{\infty} x\,P_n(x)\,t^n + \sum_{n=0}^{\infty} P_n(x)\,t^{n+1} = 0$$

Change index in Σ's so as t^n is the power of t in all of them

$$\sum_{n=0}^{\infty}(n+1)\,P_{n+1}(x)\,t^n + \sum_{n=1}^{\infty} -2xn\,P_n(x)\,t^n + \sum_{n=2}^{\infty}(n-1)\,P_{n-1}(x)\,t^n$$

$$- \sum_{n=0}^{\infty} x\,P_n(x)\,t^n + \sum_{n=1}^{\infty} P_{n-1}(x)\,t^n = 0$$

Combining as on Σ while taking extra terms out

$$P_1(x) + 2P_2(x)t - 2x P_1(x) t$$

$$- x P_0(x) - x P_1(x) t + P_0(x)$$

$$+ \sum_{n=2}^{\infty} \{(n+1) P_{n+1} - (2n+1) x P_n + n P_{n-1}\} t^n = 0$$

coeff $(t^n) = 0 \implies$

$$(n+1) P_{n+1}(x) - (2n+1)\, x P_n + n P_{n-1}(x) = 0$$

which is the required result.

<u>Example</u>. Using $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ we get obtain $P_3(x)$ from this result by putting $n = 2$.

$$3P_3(x) - 5x P_2 + 2P_1 = 0$$

$$3P_3(x) = 5x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - 2x$$

$$= \frac{15}{2}x^3 - \frac{5}{2}x - 2x = \frac{15}{2}x^2 - \frac{9}{2}x$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

<u>Norm of $P_n(x)$</u> : As we have seen that

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

where $$a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} P_n^2(x)\, dx} = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\|P_n(x)\|^2}$$

It is therefore useful to have a formula for $\|P_n(x)\|$.

Consider the Re-currence relation

$$(n+1)\, P_{n+1} + n\, P_{n-1} = (2n+1)\, x\, P_n(x) \quad \text{—— (a)}$$

$n \to n-1$ gives

$$n\, P_n + (n-1)\, P_{n-2} = (2n-1)\, x\, P_{n-1} \quad \text{—— (b)}$$

Take scalar product of (a) with P_{n-1} and of (b) with P_n

$$(n+1)\langle P_{n+1}, P_{n-1}\rangle + n\langle P_{n-1}, P_{n-1}\rangle = (2n+1)\langle x P_n, P_{n-1}\rangle$$

$$n\langle P_n, P_n\rangle + (n-1)\langle P_{n-2}, P_n\rangle = (2n-1)\langle x P_{n-1}, P_n\rangle$$

Using orthogonality $\langle P_{n+1}, P_{n-1}\rangle = 0$
$\langle P_{n-2}, P_n\rangle = 0$,

so

$$n\langle P_{n-1}, P_{n-1}\rangle = (2n+1)\langle x P_n, P_{n-1}\rangle$$

and $n\langle P_n, P_n\rangle = (2n-1)\langle x P_{n-1}, P_n\rangle$

Note that $\langle x P_n, P_{n-1}\rangle$ and $\langle x P_{n-1}, P_n\rangle$ both are equal to $\int_{-1}^{1} x\, P_n(x)\, P_{n-1}(x)\, dx$

Eliminating this we get

$$(2n+1)\,\|P_n\|^2 = (2n-1)\,\|P_{n-1}\|^2$$

From this relation we can write

$$3\|P_1\|^2 = 1\ \|P_0\|^2$$

$$5\|P_2\|^2 = 3\|P_1\|^2$$

$$\text{---}\quad\text{---}$$

$$\text{---}\quad\text{---}$$

$$(2n-1)\|P_{n-1}\|^2 = (2n-3)\|P_{n-2}\|^2$$

$$(2n+1)\|P_n\|^2 = (2n-1)\|P_{n-1}\|^2$$

Taking product and cancelling repeated terms on both sides

$$(2n+1)\|P_n\|^2 = \|P_0\|^2$$

or $$\|P_n\|^2 = \frac{\|P_0\|^2}{(2n+1)}$$

Now $$\|P_0\|^2 = \int_{-1}^{1} 1 . dx = [x]_{-1}^{1} = 2$$

Hence $$\|P_n\|^2 = \frac{2}{2n+1}$$

or $$\|P_n\| = \sqrt{\frac{2}{2n+1}}\,.$$

<u>Exercise</u> Express (a) $f(x) = \begin{cases} 1 & , -1<x<0 \\ 0 & , 0<x<1 \end{cases}$

(b) $f(x) = 1$

as Legendre series

Solution of Boundary value Problem

Let us consider steady state temperature distribution in a sphere of radius a. If radial symmetry is assumed, the temperature u is independent of ϕ and the Laplace equation in terms of spherical coordinates is

$$r\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0, \quad r<a, \quad 0<\theta<\pi \qquad (1)$$

with temperature at surface given by

$$u(a,\theta) = F(\theta), \qquad 0<\theta<\pi. \qquad (2)$$

Assume $u(r,\theta) = R(r)\,G(\theta)$, we obtain from (1)

$$\left[r\frac{\partial^2}{\partial r^2}(rR(r))\right]G(\theta) + \frac{R(r)}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, G'(\theta)\right) = 0$$

or

$$\frac{\left[r\frac{d^2}{dr^2}(rR(r))\right]}{R(r)} = -\frac{1}{\sin\theta}\,\frac{\frac{d}{d\theta}\left(\sin\theta\, G'(\theta)\right)}{G(\theta)} = \lambda$$

where λ is the separation constant.

This gives

$$r\frac{d^2}{dr^2}(rR(r)) - \lambda R = 0 \qquad (3)$$

and

$$\frac{d}{d\theta}\left(\sin\theta\frac{dG}{d\theta}\right) + \lambda\sin\theta\, G(\theta) = 0 \qquad (4)$$

In (4), we put $x = \cos\theta$, $\frac{dx}{d\theta} = -\sin\theta$

so that $\frac{dG}{d\theta} = \frac{dG}{dx}\frac{dx}{d\theta} = -\frac{dG}{dx}\sin\theta$.

hence

$$\sin\theta\frac{dG}{d\theta} = -\sin^2\theta\,\frac{dG}{dx} = -(1-\cos^2\theta)\,\frac{dG}{dx} = -(1-x^2)\,\frac{dG}{dx}$$

Also we notice $\frac{d}{d\theta} = -\sin\theta\,\frac{d}{dx}$

Thus $\frac{d}{d\theta}\left(\sin\theta\,\frac{dG}{d\theta}\right) = -\sin\theta\,\frac{d}{dx}\left[-(1-x^2)\frac{dG}{dx}\right]$

Hence (4) gives

$$\cancel{\sin\theta}\,\frac{d}{dx}\left[(1-x^2)\,\frac{dG}{dx}\right] + \lambda\,\cancel{\sin\theta}\,G(x) = 0 \qquad (5)$$

Equation (5) can be recognized as the Legendre equation. Thus $\lambda_n = n(n+1)$, $n=0,1,2,\ldots$

$G(x) = P_n(x)$ " "

or in terms of θ, $G_n(\theta) = P_n(\cos\theta)$

Equation (3) for $\lambda = n(n+1)$ gives

$$r\frac{d^2}{dr^2}(rR(r)) - n(n+1)\;R = 0 \quad \text{or}$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0$$

Using $t = \ln r$, $\frac{dt}{dr} = \frac{1}{r}$

$$\frac{dR}{dr} = \frac{dR}{dt}\frac{dt}{dR} = \frac{1}{r}\frac{dR}{dt}$$

$$\frac{d^2R}{dr^2} = \frac{1}{r^2}\frac{d^2R}{dt^2} - \frac{1}{r^2}\frac{dR}{dt}$$

This gives

$$\frac{d^2R}{dt^2} + \frac{dR}{dt} - n(n+1)R = 0$$

It has solution

$$R(r) = C_1 r^n + C_2 r^{-n-1}$$

To keep this bounded at $r=0$, we must have $C_2 = 0$. i.e. $R_n(r) = C_n r^n$. Using $u_n(r,\theta)$ and

By principle of superposition,

$$u(r,\theta) = \sum_{n=0}^{\infty} B_n r^n P_n(\cos\theta)$$

Using $u(a,\theta) = F(\theta)$, we get

$$F(\theta) = \sum_{n=0}^{\infty} B_n a^n (P_n \cos\theta)$$

The constant $B_n a^n$ can be found to be

$$B_n a^n = \frac{2n+1}{2} \int_0^{\pi} F(\theta) P_n(\cos\theta) \sin\theta \, d\theta.$$