Chapter 5

Sturm Liouville's Problems

We have seen that the separation of variables of the PDE's resulted in solving a boundary value problem involving ODE of the types

$$y'' = \lambda y \quad , 0 < x < l$$
$$y(0) = A, \quad y(l) = B.$$

In this chapter, we describe the theory of this type of problems.

We first consider the second order differential equation

$$y'' + R(x)\, y' + \{Q(x) + \lambda P(x)\} y = 0 \quad , a \le x \le b \quad \text{— (1)}$$

By using the transformation

$$r(x) = e^{\int R(x)\,dx},$$ we shall put (1) in a suitable form. For this, multiply (1) by $r(x)$

$$y'' e^{\int R(x)\,dx} + R(x)\, y' e^{\int R(x)\,dx} + \{Q(x) + \lambda P(x)\} e^{\int R(x)\,dx} y = 0$$

If $r(x) \neq 0$, this new ODE has the same solutions as (1).

Put $r(x) = e^{\int R(x)\,dx}$; $r(x)\, Q(x) = q(x)$ and $r(x)\, P(x) = p(x)$, we then have

$$r(x)\, y'' + r'(x)\, y' + \{q(x) + \lambda p(x)\} y = 0$$

or $$(r(x)\, y')' + \{q(x) + \lambda p(x)\} y = 0 \quad \text{— (2)}$$
$$a \le x \le b$$

Equation (2) is called the Sturm-Liouville form of Sturm-Liouville diff. equation.

Example: ① Consider $y'' + \frac{2}{x}y' + (x^2 - \lambda x)y = 0$

Here $R(x) = \frac{2}{x}$, $e^{\int R(x)dx} = e^{\int 2/x \, dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiply by x^2,

$$x^2 y'' + 2x y' + (x^4 - \lambda x^3) y = 0$$

or $(x^2 y')' + (x^4 - \lambda x^3) y = 0$

which is the S-L form.

The Cauchy Euler equation is

$$x^2 y'' + x y' + \lambda y = 0$$

In the form ①, we have

$$y'' + \frac{1}{x} y' + \frac{\lambda}{x^2} y = 0 \quad \text{---} *$$

So $R(x) = \frac{1}{x}$, $e^{\int R(x)dx} = e^{\int \frac{1}{x} dx} = e^{\int \ln x} = x$

Hence we multiply (*) by x,

$$x y'' + y' + \frac{\lambda}{x} y = 0,$$

which can be written as

$$(x y')' + \frac{\lambda}{x} y = 0$$

which is the S-L form.

3) The Legendre equation is

$$(1 - x^2) y'' - 2 x y' + \lambda y = 0$$

SL form is $((1-x^2)y')' + \lambda y = 0$ ✓

1) The Bessel equation

$x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0$ has the SL form $(xy')' + (\lambda x - \frac{\nu^2}{x})y = 0$.

Regular Sturm Liouville Problem:

$$(ry')' + (q + \lambda p)y = 0 \quad , a < x < b \qquad (3)$$

$$\left.\begin{aligned} a_1 y(a) + a_2 y'(a) &= 0 \quad (i) \\ b_1 y(b) + b_2 y'(b) &= 0 \quad (ii) \end{aligned}\right\} \qquad (3a)$$

provided that

(i) p, q, r and r' are continuous on the closed interval $a \le x \le b$

(ii) $p(x) > 0$ and $r(x) > 0$, $a \le x \le b$.

Periodic Sturm Liouville Problem

In (3a), we assume $r(a) = r(b)$,

$$y(a) = y(b), \quad y'(a) = y'(b) \qquad (3b)$$

Singular Sturm - Liouville Problem

Type 1 $\quad r(a) = 0, \quad b_1 y(b) + b_2 y'(b) = 0$

(Boundary condition (i) from 3 (a) is dropped)

Type 2 $\quad r(b) = 0, \quad a_1 y(a) + a_2 y'(a) = 0$

(Boundary condition (ii) from (3a) is dropped)

Type 3. $r(a) = r(b) = 0$

No boundary condition is specified in this case.

Sturm Liouville's Theorem

(i) There exists an infinite number of real eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$

(ii) The ~~corresponding~~ ~~eigenfunctions~~ eigenvalues are simple. This means corresponding to each eigenvalue there is only one linearly independent eigenfunction

(iii) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with weight function $p(x)$ i.e

$$\int_a^b p(x)\, y_m(x)\, y_n(x)\, dx = 0 \quad , m \neq n. \quad (\lambda_m \neq \lambda_n).$$

Examples

① $y'' + \lambda y = 0$, $0 < x < 1$, $\quad y(0) = 0, \quad y(1) + y'(1) = 0$

② Consider

$$(x y'(x))' + \frac{\lambda}{x} y(x) = 0 \quad , 1 < x < b$$

$$y(1) = 0, \quad y(b) = 0.$$

The D.E is in fact the Cauchy-Euler equation

$$x^2 y''(x) + x y'(x) + \lambda y(x) = 0.$$

Ans: $\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{(\ln b)^2}$ $\cancel{\frac{(2n-1)^2\pi^2}{4 \ln b}}$, $\phi_n(x) = \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln x)$

$n = 1, 2, 3 \ldots$

Solution :

(2) Write the D. E as

$$x^2 y''(x) + x\, y'(x) + \lambda\, y(x) = 0$$

Put $t = \ln x$ ($x = e^t$) , $\frac{dt}{dx} = \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

So, we get

$$\frac{d^2y}{dt^2} - \cancel{\frac{dy}{dt}} + \cancel{\frac{dy}{dt}} + \lambda y = 0$$

The auxiliary eqn : $m^2 + \lambda = 0$, $m = \pm i\sqrt{\lambda}$

$y(t) = A \cos\sqrt{\lambda}\, t + B \sin\sqrt{\lambda}\, t$

$y(1) = 0$, $y(b) = 0$ in terms of x can be transformed

you using $t = \ln x$ to be

$y(0) = 0$, $y(\ln b) = 0$.

Applying $y(0) = 0$, $A = 0$.

$y(\ln b) = 0 \Rightarrow B \sin(\sqrt{\lambda}\ln b) = 0$

$B \neq 0$, so $\sqrt{\lambda}\ln b = n\pi$, $n = 1, 2, 3, -$

n $\sqrt{\lambda} = \frac{n\pi}{\ln b}$

The corresponding eigenfunction

$\sin\left(\frac{n\pi}{\ln b} t\right)$ which can be normalized

by choosing $B = \frac{1}{\left\|\sin\left(\frac{n\pi}{\ln b} t\right)\right\|} = 1\Big/\left\{\int_0^{\ln b} \sin^2 \frac{n\pi}{\ln b} t \, dt\right\}^{1/2}$

$$= \sqrt{\frac{2}{\ln b}}\,.$$