

Sturm Liouville's Problems

We have seen that the separation of variables of the PDE's resulted in solving a boundary value problem involving ODE of the type

$$y'' = \lambda y, \quad 0 < x < l$$

$$y(0) = A, \quad y(l) = B.$$

In this chapter, we describe the theory of this type of problems.

We first consider the second order differential equation

$$y'' + R(x)y' + \{Q(x) + \lambda P(x)\}y = 0, \quad a \leq x \leq b \quad \text{--- (1)}$$

By using the transformation

$$r(x) = e^{\int R(x) dx}, \quad \text{we shall put (1) in a}$$

suitable form. For this, multiply (1) by $r(x)$

$$y'' e^{\int R(x) dx} + R(x)y' e^{\int R(x) dx} + \{Q(x) + \lambda P(x)\} e^{\int R(x) dx} y = 0$$

If $r(x) \neq 0$, this new ODE has the same solutions as (1).

Put $r(x) = e^{\int R(x) dx}$; $r(x)Q(x) = q(x)$ and

$r(x)P(x) = p(x)$, we then have

$$r(x)y'' + r'(x)y' + \{q(x) + \lambda p(x)\}y = 0$$

$$\text{or } (r(x)y')' + \{q(x) + \lambda p(x)\}y = 0 \quad \text{--- (2)}$$

$$a \leq x \leq b$$

Equation (2) is called the Sturm-Liouville form
of Sturm-Liouville diff. equation.

Example: ① Consider $y'' + \frac{2}{x}y' + (x^2 - \lambda x)y = 0$

Here $R(x) = \frac{2}{x}$, $\int R(x)dx = \int \frac{2}{x} dx = 2 \ln x = \ln x^2 = x^2$

Multiply by x^2 ,

$$x^2 y'' + 2x y' + (x^4 - \lambda x^3) y = 0$$

$$\text{or } (x^2 y')' + (x^4 - \lambda x^3) y = 0$$

which is the S-L form.

The Cauchy Euler equation is

$$x^2 y'' + x y' + \lambda y = 0$$

In the form ①, we have

$$y'' + \frac{1}{x}y' + \frac{\lambda}{x^2}y = 0 \quad \text{---*}$$

So $R(x) = \frac{1}{x}$, $\int R(x)dx = \int \frac{1}{x} dx = \ln x = x$

Hence we multiply (*) by x ,

$$x y'' + y' + \frac{\lambda}{x} y = 0,$$

which can be written as

$$(x y')' + \frac{\lambda}{x} y = 0$$

which is the S-L form.

3) The Legendre equation is

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

SL form is $((1-x^2)y')' + \lambda y = 0$ ✓

D) The Bessel equation

$$x^2 y'' + x y' + (\lambda x^2 - \nu^2) y = 0 \text{ has the}$$

SL form

$$(x y')' + (\lambda x - \frac{\nu^2}{x}) y = 0.$$

Regular Sturm Liouville Problem,

$$(r y')' + (q + \lambda p) y = 0, \quad a < x < b \quad (3)$$

$$\left. \begin{aligned} a_1 y(a) + a_2 y'(a) &= 0 & (i) \\ b_1 y(b) + b_2 y'(b) &= 0 & (ii) \end{aligned} \right\} (3a)$$

provided that

- (i) p, q, r and r' are continuous on the closed interval $a \leq x \leq b$
- (ii) $p(x) > 0$ and $r(x) > 0, \quad a \leq x \leq b.$

Periodic Sturm Liouville Problem

In (3a), we assume $r(a) = r(b),$
 $y(a) = y(b), \quad y'(a) = y'(b) \quad (3b)$

Singular Sturm - Liouville Problem

Type 1 $r(a) = 0, \quad b_1 y(b) + b_2 y'(b) = 0$
 (Boundary condition (i) from 3 (a) is dropped)

Type 2 $r(b) = 0, \quad a_1 y(a) + a_2 y'(a) = 0$
 (Boundary condition (ii) from (3a) is dropped)

Type 3: $y(a) = y(b) = 0$

No boundary condition is specified in this case.

Sturm Liouville's Theorem

- (i) There exists an infinite number of real eigenvalues $\lambda_1 < \lambda_2 < \dots$ with $\lim_{n \rightarrow \infty} \lambda_n = \infty$
- (ii) The ~~corresponding eigenfunctions~~ ^{eigenvalues} are simple. This means corresponding to each eigenvalue there is only one linearly independent eigenfunction
- (iii) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with weight function $p(x)$ i.e.

$$\int_a^b p(x) y_m(x) y_n(x) dx = 0, \quad m \neq n. \quad (\lambda_m \neq \lambda_n).$$

Examples

① $y'' + \lambda y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) + y'(1) = 0$

② Consider $(x y'(x))' + \frac{\lambda}{x} y(x) = 0, \quad 1 < x < b$
 $y(1) = 0, \quad y(b) = 0.$

The D.E is in fact the Cauchy-Euler equation

$$x^2 y''(x) + x y'(x) + \lambda y(x) = 0.$$

Ans: $\lambda_n = \alpha_n^2 = \frac{n^2 \pi^2}{4 \ln^2 b}, \quad \phi_n(x) = \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln x)$
 $n = 1, 2, 3, \dots$

Solution:

(46)

(2) Write the D.E as

$$x^2 y''(x) + x y'(x) + \lambda y(x) = 0$$

Put $t = \ln x$ ($x = e^t$), $\frac{dt}{dx} = \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

$$\frac{d^2 y}{dx^2} = \frac{1}{x^2} \frac{d^2 y}{dt^2} - \frac{1}{x^2} \frac{dy}{dt}$$

So, we get

$$\frac{d^2 y}{dt^2} - \frac{dy}{dt} + \lambda y = 0$$

The auxiliary eqn: $m^2 + \lambda = 0$, $m = \pm i\sqrt{\lambda}$

$$y(t) = A \cos \sqrt{\lambda} t + B \sin \sqrt{\lambda} t$$

$y(1) = 0$, $y(b) = 0$ in terms of x can be transformed
using $t = \ln x$ to be

$$y(0) = 0, \quad y(\ln b) = 0.$$

Applying $y(0) = 0$, $A = 0$.

$$y(\ln b) = 0 \Rightarrow B \sin(\sqrt{\lambda} \ln b) = 0$$

$B \neq 0$, so $\sqrt{\lambda} \ln b = n\pi$, $n = 1, 2, 3, \dots$

$$n \sqrt{\lambda} = \frac{n\pi}{\ln b}$$

The corresponding eigenfunction

$\sin\left(\frac{n\pi t}{\ln b}\right)$ which can be normalized

by choosing $B = \frac{1}{\left\| \sin\left(\frac{n\pi t}{\ln b}\right) \right\|} = \frac{1}{\left\{ \int_0^{\ln b} \sin^2 \frac{n\pi t}{\ln b} dt \right\}^{1/2}}$

$$= \sqrt{\frac{2}{\ln b}}$$