Chapter 2

Separation of Variables — Fourier Method

Linear and Homogeneous PDE

We consider the boundary value problems involving the pde's of the type

$$L u = 0 \quad \text{——— (1)}$$

where L is linear in the sense

$$L(\alpha u_1 + \beta u_2) = \alpha L u_1 + \beta L u_2 .$$

An example can be

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0$$

$A, B, \dots, F$ constants or functions of independent variables only.

Principle of superposition.

If $u_1, u_2, \dots u_n$ satisfy PDE 1 then so does $c_1 u_1 + c_2 u_2 + \dots + c_n u_n$.

$c_1, c_2, \dots, c_n$ constants.

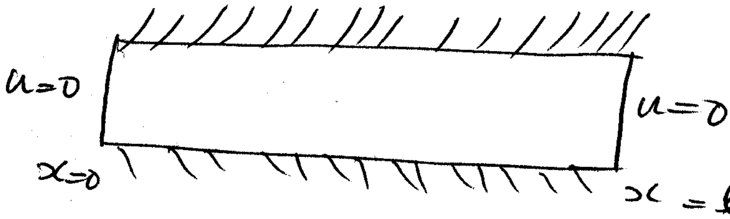
This can be generalized to $c_1 u_1 + c_2 u_2 + \dots\dots$ where the infinite series $\sum_{k=1}^{\infty} c_k u_k$ is assumed to be convergent and differentiable term by term as many times as is needed in the definition of L.

Heat Flow Problem

Let us consider a thin rod or wire made of homogeneous material (so the material constants are constants)

Assuming that the two ends are kept at zero temperature and the long edges are insulated (no heat loss), (1

the problem gives rise to



$$k u_{xx} = u_t \quad , \quad 0 < x < l \, , \, t > 0 \qquad \text{—— (2)}$$

$$u(0,t) = 0 \, , \quad u(l,t) = 0 \quad , \quad t > 0 \qquad \text{— (3)}$$ (Boundary Conditions)

$$u(x,0) = f(x) \quad , \quad 0 < x < l \qquad \text{— (4)}$$ (Initial condition

Method of solution

~~Step~~ Assume $u(x,t) = F(x)\, G(t)$

Put in (2) and divide throughout by $F(x)\, G(t)$

$$\frac{F''(x)}{F(x)} = \frac{1}{k} \frac{G'(t)}{G(t)} = \lambda \text{ say} \qquad \text{—— (5)}$$

(Separation of variables argument!)

There are three cases here.

$\lambda = 0$ This leads to $\dfrac{F''(x)}{F(x)} = 0$ or

$$F(x) = Ax + B$$

$$\boxed{\begin{aligned} \text{Using } u(0,t) &= F(0)\, G(t) = 0 \implies F(0) = 0 \\ u(l,t) &= F(l)\, G(t) = 0 \implies F(l) = 0 \end{aligned}}$$

we get $A = B = 0$. This leads to the trivial solution $u(x,y) = 0$. This is therefore rejected.

$\lambda = +\text{ve}$: This gives $F''(x) = p^2 F(x)$, $\lambda = p^2 > 0$

This gives $F(x) = A e^{px} + B e^{-px}$

$F(0) = 0 \Rightarrow \quad A + B = 0$ — (6)

$F(l) = 0 \Rightarrow \quad A e^{pl} + B e^{-pl} = 0$ — (7)

(6) and (7) are homogeneous system in A, B. For non-trivial solution $\quad \det(\text{coeff}) = 0 \Rightarrow$

$$\begin{vmatrix} 1 & 1 \\ e^{pl} & e^{-pl} \end{vmatrix} = 0 \Rightarrow e^{-pl} = e^{pl} \Rightarrow p = 0$$

which gives $\lambda = 0$, already considered.

$\underline{\lambda = -ve}$: $\quad \lambda = -p^2$

This corresponds to $\quad F''(x) + p^2 F(x) = 0$

which has solution $\quad F(x) = A \cos px + B \sin px$

$F(0) = 0 \Rightarrow \quad A = 0$

$F(l) = 0 \Rightarrow \quad B \sin pl = 0$

$B \neq 0$ (otherwise we get trivial solution)

gives $\sin pl = 0 \Rightarrow pl = n\pi$ or $p = \frac{n\pi}{l}$

$n = 1, 2, 3, \ldots$

so $\lambda = -\frac{n^2\pi^2}{l^2}$ (These values of λ are called eigenvalues — they correspond to non-trivial solutions of $F'' + p^2 F(x) =$

Thus, we lable λ corresponding to $n = 1, 2, 3,$ as

$$\lambda_n = -\frac{n^2\pi^2}{l^2}, \quad n = 1, 2, 3, \ldots \quad \text{and} \quad \text{—— (8)}$$

$$\boxed{F_n(x) = B_n \sin \frac{n\pi}{l} x, \quad n = 1, 2, 3, \ldots} \quad \text{—— (9)}$$

$\underline{\text{Equation in } t}$: The second D.E. from (5) is

$$G'(t) = \lambda k G(t)$$

which gives $\quad G(t) = C e^{\lambda k t}$

As $\lambda = \lambda_n$, given by (8)

$$G_n(t) = C_n\, e^{-\frac{n^2\pi^2}{l^2}kt}, \qquad n=1,2,\ldots \qquad (10)$$

The product of (9) and (10) gives

$$u_n(x,t) = E_n\, e^{-\frac{n^2\pi^2}{l^2}kt} \sin\frac{n\pi x}{l}, \quad n=1,2,\ldots \qquad (11)$$

We now note that we cannot, in general satisfy the initical conditions (4). By principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} E_n\, e^{-\frac{n^2\pi^2}{l^2}kt} \sin\frac{n\pi x}{l}$$

$u(x,0)=f(x)$ gives

$$\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} = f(x)$$

Multiplying by $\sin\frac{m\pi x}{l}$ and integrating from 0 to l

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}, \quad n=1,2,3\ldots$$

This completes the solution of the problem.

<u>Remark</u>: Notice that

$$\lim_{t\to\infty} u(x,t) = 0$$

as should be expected from physical consideration.

Example Solve

$$u_{xx} = \frac{1}{k} u_t \,, \quad 0<x<\pi,\ t>0 \quad (k \text{ constant})$$

$$u(0,t)=0, \quad u(\pi,t)=0, \quad t>0$$

$$u(x,0)= T_0 \quad (\text{constant}), \quad 0<x<\pi.$$

Solution. Separation of variables

(write all steps)

$$F'' + \lambda F = 0, \quad F(0)=0, \quad F(\pi)=0 \quad \text{———} (1)$$

and $$G'(t) + \lambda k\, G(t) = 0 \quad \text{———} (2)$$

(1) can be seen to have solution

$$\lambda_n = n^2, \quad F_n(x) = \sin nx, \quad n = 1, 2, 3, \ldots$$

(2) gives $$G_n(t) = c_n e^{-kn^2 t}, \quad n = 1, 2, 3, \ldots$$

By principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)\, e^{-kn^2 t}$$

$$u(x,0) = T_0 \Rightarrow T_0 = \sum_{n=1}^{\infty} c_n \sin nx$$

From which (multiply by $\sin mx$ and integrate from 0 to π)

$$c_n = \frac{2}{\pi}\int_0^{\pi} T_0 \sin nx\, dx = \frac{2T_0}{n\pi}\left[1-(-1)^n\right] \quad n = 1, 2, 3, \ldots$$

$C_n = 0$, when n is even

$= 2$, n even.

$$u(x,t) = \frac{4T_0}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k+1)x}{k+1}\, e^{-k(2k+1)^2 t}$$

Exercise (Text Book Page 41)

$u_t = k\,u_{xx}$ (k constant), $0 < x < c$, $t > 0$

$u_x(0,t) = 0$, $u_x(c,t) = 0$, $t > 0$

$u(x,0) = f(x)$

Solution (Ans)

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 k t}{c^2}\right) \cos\frac{n\pi x}{c}$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x) \cos\frac{n\pi x}{c}\,dx, \quad n = 1, 2, 3, \ldots$$