<u>Case 1</u>: $\lambda = 0$ In this case

$$F''(x) = 0 \Rightarrow F(x) = Ax + B, \; F'(x) = A$$

$$F(0) = F(l) \Rightarrow B = Al + B$$

$$\Rightarrow Al = 0 \Rightarrow A = 0 \quad (l \neq 0)$$

$F'(0) = F'(l)$ does not give any further information.

$F(x) = B$ corresponds to $\lambda = 0$.

<u>Case 2</u> $\lambda > 0$. In this case we take $\lambda = k^2$, $k$ real. Then

$$\frac{F''(x)}{F(x)} = k^2 \quad \text{or} \quad F''(x) - k^2 F(x) = 0$$

$m^2 - k^2 = 0$, $m = \pm k$ and so

$$F(x) = c_1 e^{kx} + c_2 e^{-kx}$$

$F(0) = F(l)$ gives

$$c_1 + c_2 = c_1 e^{kl} + c_2 e^{-kl}$$

$F'(0) = F'(l)$ gives

$$\cancel{k} c_1 - \cancel{k} c_2 = \cancel{k} c_1 e^{kl} - \cancel{k} c_2 e^{-kl}$$

These two equations can be written as

$$c_1(1 - e^{kl}) + c_2(1 - e^{-kl}) = 0 \quad \text{—(5)}$$

and

$$c_1(1 - e^{kl}) - c_2(1 - e^{-kl}) = 0 \quad \text{—(6)}$$

(5) and (6) can have non-trivial soluti[ons] only if $K = 0$ (case 1)

Wave Equation, Vibration of String

PDE: $u_{xx} = \frac{1}{c^2} u_{tt}, \quad 0<x<l, t>0$ (1)

Boundary Conditions: $\left.\begin{aligned} u(0,t) &= u(l,t), t>0 \\ u_x(0,t) &= u_x(l,t), t>0 \end{aligned}\right\}$ Periodic B.C's. (2a)

Initial Conditions: $\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{aligned}\right\} 0<x<l$ (2b)

[For variety, we have chosen to solve the wave equation under a different set of boundary conditions]

Solution

I Let us assume $u(x,t) = F(x)G(t)$

Then $u_{xx} = F''(x)G(t)$

$u_{tt} = F(x)G''(t)$

PDE $\Rightarrow F''(x)G(t) = \frac{1}{c^2}F(x)G''(t)$

or $\frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''(t)}{G(t)} = \lambda$ (say) (3)

Boundary Conditions $\Rightarrow u(0,t) = u(l,t) \Rightarrow$

$$\left.\begin{aligned} &F(0)G(t) = F(l)G(t) \Rightarrow F(0) = F(l) \\ &u_x(0,t) = u_x(l,t) \Rightarrow F'(0)G(t) = F'(l)G(t) \\ &\Rightarrow F'(0) = F'(l) \end{aligned}\right\} (4)$$

($\because$ As $G(t) \neq 0$)

Case 3 $\lambda = -k^2$. $(\lambda < 0)$

In this ~~w~~ case we obtain

$F''(x) + k^2 F(x) = 0$ corresponding auxiliary equation has roots $m = \pm ik$ which gives

$F(x) = A \cos kx + B \sin kx$

$F'(x) = -Ak \sin kx + Bk \cos kx$

$F(0) = F(l) \Rightarrow A = A \cos kl + B \sin kl$

or $(1 - \cos kl) A - B \sin kl = 0$ —(7)

$F'(0) = F'(l) \Rightarrow -Ak \sin kl + Bk \cos kl = B\cancel{k}$

or $(-\sin kl) A + (\cos kl - 1) B = 0$ —(8)

For non trivial solutions of (7) and (8)

$$\begin{vmatrix} 1 - \cos kl & -\sin kl \\ -\sin kl & \cos kl - 1 \end{vmatrix} = 0 \Rightarrow -(1 - \cos kl)^2 - \sin^2 kl = 0$$

$\Rightarrow -1 - \cos^2 kl + 2 \cos kl - \sin^2 kl = 0$

or $\cos kl = 1 \Rightarrow kl = 2n\pi, \; n = 1, 2, 3, \ldots$

or $k_n = \frac{2n\pi}{l}, \; n = 1, 2, 3, \ldots$ (labeled with

[If we combine case (1) case with it $(k = 0)$, we may write $k_n = \frac{2n\pi}{l}, \; n = 0, 1, 2, 3, \ldots$]

$F_n(x) = A_n \cos \frac{2n\pi}{l} x + B_n \sin \frac{2n\pi}{l} x.$

Choosing $A_n = 0, B_n = 1$ and then $A_n = 1, B_n = 0$ we get two eigenfunctions corresponding to $k_n = \frac{2n\pi}{l}$

$\sin \frac{2n\pi}{l} x$ and $\cos \frac{2n\pi}{l} x$.

We can either write

$$F_0 = 1, \quad F_n = A_n \cos\frac{2n\pi}{\ell}x + B_n \sin\frac{2n\pi x}{\ell}$$

$n = 1, 2, 3 \ldots$

or $F_n = A_n \cos\frac{2n\pi x}{\ell} + B_n \sin\frac{2n\pi x}{\ell}$, $n = 0, 1, 2, 3 \ldots$

Usually first way of writing is preferred.

II. Corresponding to $\lambda_n = \frac{2n\pi}{\ell}$, we get

$$G_n''(t) + \frac{4n^2\pi^2}{\ell^2} G(t) = 0 \qquad \text{giving us}$$

$$G_n(t) = C_n \cos\frac{2n\pi}{\ell}t + D_n \sin\frac{2n\pi}{\ell}t$$

$$G_n'(t) = -\frac{2n\pi}{\ell} C_n \sin\frac{2n\pi}{\ell}t + \frac{2n\pi}{\ell} D_n \cos\frac{2n\pi}{\ell}t$$

$$G_n'(t) = 0 \Rightarrow D_n = 0$$

Thus $G_n(t) = C_n \cos\frac{2n\pi}{\ell}t$

$u_n = F_n(x)\, G_n(t)$ gives

$$u_n(x,t) = \left(A_n \cos\frac{2n\pi x}{\ell} + B_n \sin\frac{2n\pi x}{\ell}\right) C_n \cos\frac{2n\pi}{\ell}t$$

$n = 0, 1, 2, \ldots$

so

$$\sum_{n=0}^{\infty} u_n(x,t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{\ell} + b_n \sin\frac{2n\pi x}{\ell}\right)\cos\frac{2n\pi}{\ell}t$$

( $n=0$ value have been separated from $\Sigma$ )

We can now determine $a_0, a_n, b_n$ using the remaining initial condition and integration

III Using $u_n(x,0) = f(x)$ we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi}{l}x + b_n \sin\frac{2n\pi}{l}x \qquad (10)$$

Integrating it from 0 to $l$

$$\int_0^l f(x)\,dx = a_0 \int_0^l dx + \sum_{n=1}^{\infty}\left\{a_n \int_0^l \cos\frac{2n\pi}{l}x\,dx + b_n \int_0^l \sin\frac{2n\pi}{l}x\,dx\right\}$$

or $$a_0 = \frac{1}{l}\int_0^l f(x)\,dx \qquad (11)$$

(other integrals are zero)

Multiply (10) by $\cos\frac{2m\pi x}{l}$ and integrate from 0 to $l$ (term by term inside $\Sigma$). All integrals are zero except $a_m$ ($n \Leftrightarrow m$) which can give us

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{2n\pi}{l}x\,dx$$

Similarly $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{2n\pi x}{l}\,dx.$$

This completes the solution.

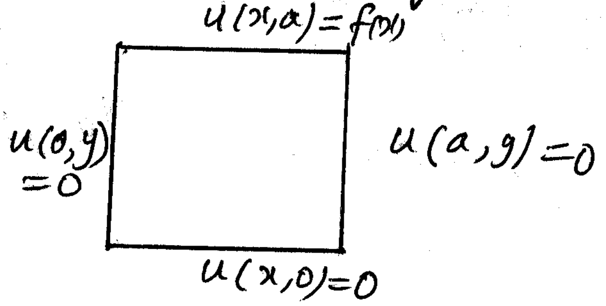
Page (24) Laplace Equation:

Steady State Heat Eqn in a plate

Consider a plate $0 \le x \le a$, $0 \le y \le a$.

When no sources are present, steady state heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$




We assume that three sides of the plate ($x=0, x=a$ and $y=0$) are kept at zero temperature. The temperature at $y=a$ is assumed to be prescribed.

$u(0,y) = 0$ , $0 \le y \le a$

$u(a,y) = 0$ , $0 \le y \le a$

$u(x,0) = 0$ , $0 < x \le a$

$u(x,a) = f(x)$ $\quad 0 < x \le a.$

Assume $u(x,y) = F(x)\,G(y)$

$u_{xx} = F''(x)\,G(y)$

$u_{yy} = F(x)\,G''(y)$

PDE $\Rightarrow$ $F''(x)\,G(y) = -F(x)\,G''(y)$

or $$\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} = \lambda \text{ (say)}$$

$\lambda = 0$ case can bee seen to lead to trivial case. We shall consider $\lambda = -k^2$ (negative).

This gives $\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -k^2$

Boundary conditions give

$$u(0,y) = F(0)\,G(y) = 0 \Rightarrow F(0) = 0$$
$$u(a,y) = F(a)\,G(y) = 0 \Rightarrow F(a) = 0$$
$$\left.\right\} G(y) \neq 0$$

$$u(x,0) = 0 \Rightarrow F(x)\,G(0) = 0 \Rightarrow G(0) = 0.$$

Now $\frac{F''(x)}{F(x)} = -k^2$ gives

$$F''(x) + k^2 F(x)$$

which leads to

$$F(x) = C \cos kx + D \sin kx$$

$F(0) = 0 \Rightarrow C = 0$

$F(a) = 0 \Rightarrow D \sin ka = 0$

For $D \neq 0$, $\sin ka = 0 \Rightarrow ka = n\pi$

or $k = \frac{n\pi}{a}$, $n = 1, 2, 3$

We write $k_n = \frac{n\pi}{a}$, $n = 1, 2, 3 \ldots$

Correspondingly

$$F_n = D_n \sin \frac{n\pi}{a} x, \quad n = 1, 2, 3 \ldots$$

II G-Equation:

$$\cancel{-}\frac{G''(y)}{G(y)} = \cancel{-}\frac{n^2\pi^2}{a^2} \quad \left(\text{As } k = \frac{n\pi}{a}\right)$$

or $G''(y) - \frac{n^2\pi^2}{a^2} = 0$

The auxiliary equation gives

$$m = \pm \frac{n\pi}{a}.$$

We write the general solution as

$$G_n(y) = E_n \sinh\frac{n\pi}{a}y + F_n \cosh\frac{n\pi}{a}y$$

($n$ due to dependence on $n$)

$$u(x,0) = 0 \implies F(x)\,G(0) = 0 \implies G(0) = 0 \quad (F(x) \neq 0)$$

Thus we get $F_n = 0$

$$G_n(y) = E_n \sinh\frac{n\pi}{a}y.$$

$$u_n(x,y) = D_n E_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y \qquad n = 1, 2, 3, \ldots$$

By superposition principle

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} M_n \sin\frac{n\pi}{a}x \sin\frac{n\pi}{a}y$$

$$(M_n = D_n E_n)$$

III Using $u(x,0) = f(x)$ we get

$$f(x) = \sum_{n=1}^{\infty} M_n \sin\frac{n\pi}{a}x$$

To get $M_n$, multiply by $\sin\frac{m\pi}{a}x$ and integrate, then change $m$ to $n$

$$M_n = \int_0^a f(x) \sin\frac{n\pi x}{a}\,dx.$$