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Acoustic waves in a layered inhomogeneous ocean

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Abstract

We consider a layered ocean of finite depth in which the lower layer is assumed to have depth dependent properties. The seabed is considered to be either rigid or of a reflecting type. The perturbation method is used to obtain the eigenvalues and the eigenfunctions of the depth equation in the case of both the rigid and reflecting seabed. The corrections to the eigenvalues and eigenfunctions are numerically computed from the perturbation formulae in some case of interest. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

The study of acoustic wave propagation in the ocean has attracted considerable attention in the past. This has been motivated by the need to understand naval detections and marine seismology. One of the earliest models based upon wave-theoretical solutions in a homogeneous model of an ocean was forwarded by Pekeris [8] who used the normal mode solution of the wave equation to explain the propagation of explosively generated sound on shallow water. Spectral representation of the wave equation was obtained under the assumption of cylindrical symmetry and using the separation of the solution of the wave equation into the so-called range and depth equations. Due to a rich source of mathematical theory available to follow this approach, various studies have followed the Pekeris wave guide model. The normal mode solution approach following Pekeris can be found in Ahluwalia [1], Boyles [2] and to some extent in Brekhovskikh and Lysanov [4], and De Santo [5].

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The homogeneous model of the ocean gives rise to a nice eigenvalue problem and the solution is known as an infinite series of eigenvalues and a family of orthonormal eigenfunctions. The eigenvalue of the problem corresponds to the wave number, and thus to the sound speed. It is however, well known [2,3] that the sound speed varies due to variation in the temperature, salinity and with the increase in depth. This variation of the sound speed, however small, significantly affects the propagation of sound in the ocean. Duston, Verma and Wood [6] have studied the inhomogeneous model of the ocean in which the sound speed, and hence the refractive index varies slightly with the depth. They used the perturbation method to obtain the solution of the depth equation in this interesting case. However, some studies [7] show that the physical characteristics of the ocean do not vary continuously or smoothly with depth, but vary in a discontinuous way. They remain piece-wise constant within certain layers and the change across the interface of these layers is rather rapid. The thickness of these layers may be variable in different situations but the horizontal extent of such layers may reach up to tens of kilometers [4]. Boyles [2] has described the layered model of the ocean to account for such a behavior.

In most studies [2,3,6] the bottom of the sea is conveniently assumed to be rigid so that the depth equation of the resulting problem has a nice behaviour. In practical situations, however, the seabed is not rigid but satisfies the absorbing or impedance boundary conditions. Physically, it means that part of the acoustic field is reflected while a part of it is absorbed into the seabed. This interesting situation is one of the main foci of this study.

In this paper, we attempt to study the problem of acoustic wave propagation in an inhomogeneous sea consisting of two layers. We use perturbation technique to study the eigenvalue problem corresponding to this case. Our treatment of the case of a single layer differs from that presented by Duston, Verma and Wood [6] in two ways. Firstly, we consider the layered model of the ocean in which the upper layer has constant physical characteristics while the lower layer has depth dependent properties. Secondly, we consider the case of reflecting seabed satisfying impedance type boundary conditions resulting from a more realistic situation in which the bottom of the sea reflects back a part of the energy as well as that of the more strict rigid boundary condition.

2. Mathematical model

We present here the two layer model of the ocean as discussed by Boyles [2]. Geometry of a such a model is shown in Fig. 1.

We consider the ocean of depth h consisting of two layers of uniform depth d so that the lower layer has depth $h - d$. The seabed is considered to be at $z = h$. The acoustic pressure $p^{(i)}$, density $\rho^{(i)}$, velocity c_i and the wave number k_{i} refer to these quantities in the ith layer, $i = 1, 2$. The Helmholtz equation satisfied by the acoustic pressure $p^{(i)}$ in the ith layer in terms of cylindrical coordinate assuming radial symmetry can be written as

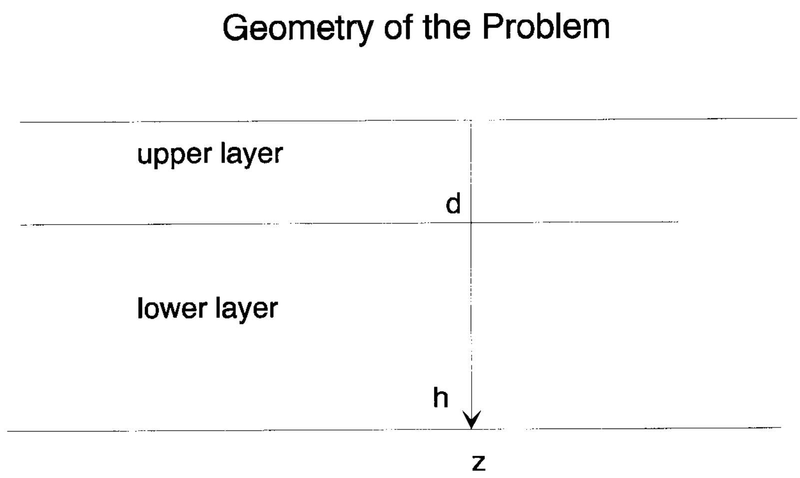

Fig. 1. Geometry of the problem.

$$\frac{\partial^2 p^{(i)}}{\partial r^2} + \frac{1}{r}\frac{\partial p^{(i)}}{\partial r} + \frac{\partial^2 p^{(i)}}{\partial z^2} + k_i^2 p^{(i)} = 0, \quad k_{\mathrm{i}} = \frac{\omega}{c_{\mathrm{i}}}\, i = 1, 2 \tag{1}$$

Using the separation of variables, $p^{(i)} = R(r)\phi^{(i)}(z)$, the depth equation for the function $\phi^{(i)}(z)$ takes the form

$$\frac{\mathrm{d}^2\phi^{(i)}}{\mathrm{d}z^2} + \left(k_i^2 - \lambda\right)\phi^{(i)} = 0, \quad i = 1, 2 \tag{2}$$

2.1. Rigid seabed

In case the seabed is assumed to be rigid, we have the following boundary conditions

(a) free surface at $z = 0$ gives

$$\phi^{(1)}(0) = 0. \tag{3}$$

(b) continuity of acoustic pressure at the interface $z = d$ gives

$$\phi^{(1)}(d) = \phi^{(2)}(d). \tag{4}$$

(c) continuity of the gradient of acoustic pressure at the interfaces give

$$\frac{1}{\rho_1}\frac{\mathrm{d}\phi^{(1)}(d)}{dz} = \frac{1}{\rho_2}\frac{\mathrm{d}\phi^{(2)}(d)}{dz}. \tag{5}$$

(d) rigid bottom at seabed $z = h$ gives

$$\frac{d\phi^{(2)}(h)}{dz} = 0. \tag{6}$$

For convenience, let

$$\gamma_i^2 = k_i^2 - \lambda,\ i = 1, 2. \tag{7}$$

The general solution of the depth Eq. (2) is given by

$$\phi^{(1)} = A\sin\gamma_1 z + D\cos\gamma_1 z,\ 0\leqslant z\leqslant d \tag{8}$$

$$\phi^{(2)} = B\sin\gamma_2 z + C\cos\gamma_2 z,\ d\leqslant z\leqslant h \tag{9}$$

Using the boundary conditions (3)–(6), we obtain

$$\begin{array}{lcl} D & = & 0, \\ A\sin\gamma_1 d & = & C\sin\gamma_2 d + B\cos\gamma_2 d, \\ \rho_2\gamma_1 A\cos\gamma_1 d & = & \rho_1\gamma_2 C\cos\gamma_2 d - \rho_1\gamma_2 B\sin\gamma_2 d, \\ \gamma_2 C\cos\gamma_2 h - \gamma_2 B\sin\gamma_2 h & = & 0. \end{array} \tag{10}$$

This constitutes a homogeneous system of equations in A, B, C and D For a non-trivial solution, we must have determinant of the coefficient matrix = 0. This gives

$$\det\begin{pmatrix} \sin\gamma_1 d & -\cos\gamma_2 d & -\sin\gamma_2 d \\ \gamma_1\rho_2\cos\gamma_1 d & \gamma_2\rho_1\sin\gamma_2 d & -\gamma_2\rho_1\cos\gamma_2 d \\ 0 & -\sin\gamma_2 h & \cos\gamma_2 h \end{pmatrix} = 0 \tag{11}$$

or, upon expanding the determinant,

$$\gamma_1\rho_2\cos\gamma_1 d\cos\gamma_2(h-d) - \gamma_2\rho_1\sin\gamma_1 d\sin\gamma_2(h-d) = 0. \tag{12}$$

This is the characteristic equations satisfied by eigenvalues λ_m. The corresponding eigenfunctions $\phi_m^{(0)}(z)$ are given by

$$\begin{array}{lcl} \phi_m^{(1)}(z) & = & A\sin\gamma_1 z;\ 0 < z < d, \\ \phi_m^{(2)}(z) & = & A\dfrac{\sin\gamma_1 d}{\cos\gamma_2(h-d)}[\cos\gamma_2(h-z)];\ d < z < h. \end{array} \tag{13}$$

The normalization condition would now imply

$$\int_o^h \frac{1}{\rho_0}\phi_n^2 dz = \int_0^d \frac{1}{\rho_1}\left(\varphi_n^{(1)}\right)dz + \int_d^h \frac{1}{\rho_2}\left(\phi_n^{(2)}\right)^2 dz = 1. \tag{14}$$

Carrying out the integration involved we may obtain

$$\int_o^h \frac{1}{\rho_0}\phi_n^2 \mathrm{d}z = \frac{\tau}{\rho_1} + \frac{1}{\rho^2}\frac{\sin^2\gamma_1 d}{\cos^2\gamma_2(h-d)}\left(\frac{(h-d)}{2} + \varphi\right) = 1/G_n^2, \tag{15}$$

where $\tau = \frac{\mathrm{d}}{2} - \frac{1}{2\gamma_1}\cos\gamma_1 d \sin\gamma_1 d$ and $\varphi = \frac{1}{2\gamma_2}\sin\gamma_2(h-d)\cos\gamma_2(h-d)$. Thus choosing $A = G_n$, the eigenfunctions would satisfy the orthonormality condition

$$\int_0^h \frac{1}{\rho_0}\phi_n\phi_m \mathrm{d}z = \delta_{nm}, \tag{16}$$

2.2. *Reflecting seabed*

We now consider the problem of an ocean consisting of two homogenous layers bounded by a pressure-release surface at the top and a reflecting type bottom below. The differential equation for the depth function $\phi^{(i)}(z)$

$$\frac{\mathrm{d}^2\phi^{(i)}}{\mathrm{d}z^2} + \left(k_i^2 - \lambda\right)\phi^{(i)} = 0, \; i = 1, 2. \tag{17}$$

The boundary conditions at $z = 0$ and $z = d$ are the same as in the case of rigid seabed given by Eqs. (3), (4), and (5). The boundary condition at the reflecting bottom however now replaces Eq. (6). The new boundary condition in this case is

$$\frac{\mathrm{d}\phi^{(2)}(h)}{\mathrm{d}z} + \alpha\phi^{(2)}(h) = 0. \tag{18}$$

We notice that if the constant $\alpha = 0$, then the rigid boundary condition Eq. (6) is recovered. The constant α gives a measure of the reflectivity of the seabed.

Using the general solution given by Eqs. (10) and (11) and using these boundary conditions, we get

$$\det\begin{pmatrix} \sin\gamma_1 d & -\cos\gamma_2 d & -\sin\gamma_2 d \\ \gamma_1\rho_2\cos\gamma_1 d & \gamma_2\rho_1\sin\gamma_2 d & -\gamma_2\rho_1\cos\gamma_2 d \\ 0 & -\gamma_2\sin\gamma_2 h + \alpha\cos\gamma_2 h & \gamma_2\cos\gamma_2 h + \alpha\sin\gamma_2 h \end{pmatrix} = 0$$

This is the characteristic equation satisfied by eigenvalues λ_m

In order to determine the eigenfunctions $\phi^{(i)}$, we solve for the unknown constants A,B, C and D in the general solution (8 and 9) subject to the boundary conditions (3)–(5) and (18) as

$$D = \quad 0$$

$$B = \quad A\left[\frac{\sin\gamma_1 d}{\cos\gamma_2 d + \left(\dfrac{\gamma_2\sin\gamma_2 h - \alpha\cos\gamma_2 h}{\gamma_2\cos\gamma_2 h + \alpha\sin\gamma_2 h}\right)\sin\gamma_2 d}\right]$$

$$C = \quad A\frac{\sin\gamma_1 d(\gamma_2\sin\gamma_2 h - \alpha\cos\gamma_2 h)}{\cos\gamma_2 d\gamma_2\cos\gamma_2 h + \cos\gamma_2 d\alpha\sin\gamma_2 h + (\sin\gamma_2 d)(\gamma_2 d)(\gamma_2\sin\gamma_2 h - \alpha\cos\gamma_2 h)} \tag{19}$$

Letting $S = (\gamma_2\sin\gamma_2 h - \alpha\cos\gamma_2 h)$, $F = \cos\gamma_2 d$, $H = \sin\gamma_2 d$, $J = \sin\gamma_1 d$, the solutions of the depth Eq. (17) become

$$\phi_{0m}^{(1)}(z) = A\sin\gamma_1 z \tag{20}$$

$$\phi^{(2)} = A\sigma\frac{(\sin\gamma_2 z)(\gamma_2\vartheta - \alpha\mu) + (\cos\gamma_2 z)(\gamma_2\mu + \alpha\vartheta)}{\eta(\gamma_2\mu + \alpha\vartheta) + (\beta)(\gamma_2\vartheta - \alpha\mu)}$$

where $\sin\gamma_2 d = \beta$, $\cos\gamma_2 d = \eta$, $\sin\gamma_2 h = \vartheta$, $\cos\gamma_2 h = \mu$, and $\sin\gamma_1 d = \sigma$.The normalization condition (16) would imply,

$$\frac{\tau}{\rho_1} - \frac{1}{\rho_2}\left(\Gamma^2 k_1 + \Omega^2 k_2\right) = 1/G_n^2, \tag{21}$$

where

$$\tau = \frac{-\cos\gamma_1 d\sin\gamma_1 d + \gamma_1 d}{2\gamma_1}, \tag{22}$$

$$\Omega = \frac{\sin\gamma_1 d}{\cos\gamma_2 d + \left(\dfrac{\gamma_2\sin\gamma_2 h - \alpha\cos\gamma_2 h}{\gamma_2\cos\gamma_2 h + \alpha\sin\gamma_2 h}\right)\sin\gamma_2 d}, \tag{23}$$

$$k_1\frac{\cos\gamma_2 h\sin\gamma_2 h - \gamma_2 h - \cos\gamma_2 d\sin\gamma_2 d + \gamma_2 d}{2\gamma_2}, \tag{24}$$

$$k_2 = \frac{-\cos\gamma_2 h\sin\gamma_2 h - \gamma_2 h + \cos\gamma_2 d\sin\gamma_2 d + \gamma_2 d}{2\gamma_2}, \tag{25}$$

and

$$\Gamma = \frac{(\gamma_2\sin\gamma_2 h - \alpha\cos\gamma_2 h)\sin\gamma_1 d}{(\cos\gamma_2 d)\gamma_2\cos\gamma_2 h + (\cos\gamma_2 d)\alpha\sin\gamma_2 h + (\sin\gamma_2 d)\gamma_2\sin\gamma_2 h - (\sin\gamma_2 d)\alpha\cos\gamma_2 h}. \tag{26}$$

The eigenfunctions of the depth equation satisfy the orthonormality condition (16) if the constant $A = G_n$ is chosen subject to the (21)–(26).

2.3. Comparison

Eq. (20) gives the eigenfunctions of the depth problem for the reflecting seabed case. In order to compare the solutions for two different kind of boundary conditions considered above, we use the fact that $\alpha = 0$ for the rigid seabed. Putting this value in (20) we obtain

$$\begin{aligned}\phi^{(2)}(z) &= A\frac{\sin\gamma_1 d}{\cos\gamma_2 d\cos\gamma_2 h + \sin\gamma_2 d\sin\gamma_2 h}[\sin\gamma_2 z\sin\gamma_2 h + \cos\gamma_2 z\cos\gamma_2 h]\\ &= A\frac{\sin\gamma_1 d}{\cos\gamma_2(h-d)}[\cos\gamma_2(h-z)],\end{aligned}$$

which agrees with (13) of Boyles' model of rigid seabed.

3. Inhomogeneous layered ocean model

In this section we consider the inhomogeneous layered ocean in which the upper layer has no significant variation in the physical properties but the lower layer has depth dependent properties. This results in the refractive index and hence the velocity to be depending on depth. We use the perturbation technique (Titchmarsh [9]) to find the eigenvalues and the eigenfunctions in case of the two sets of boundary conditions discussed in the previous section.

Since we are considering the problem of an ocean consisting of two layers bounded by a pressure-release surface above and a rigid or reflecting bottom below, it is convenient to assume

$$\Psi = \begin{cases}\phi^{(1)}\text{ in layer 1, that is,} & 0\leqslant z\leqslant d\\ \phi^{(2)}\text{ in layer 2, that is,} & d\leqslant z\leqslant h\end{cases}$$

Due to the inhomogeneous layer the depth equation is now assumed to be the perturbed equation

$$\frac{d^1\Psi}{dz^2} + k^2(1+\varepsilon s(z))\Psi = \lambda_m\Psi, \tag{27}$$

where $s(z)$ determines the depth dependent inhomogeneity. If $s(z) = 0$ the problem reduces to that of homogeneous layered ocean discussed above. Following Titchmarsh [9] we assume

$$\Psi = \Psi_0 + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \cdots\cdots \tag{28}$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots\cdots \tag{29}$$

Substituting (28) and (29) in Eq. (27) we get

$$\begin{aligned}&\Psi_0'' + \varepsilon\Psi_1'' + \varepsilon^2\Psi_2'' + \cdots\cdots + k^2(1+\varepsilon s(z))\big(\Psi_0 + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \cdots\cdots\big)\\ &\quad = \big(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots\cdots\big)\big(\Psi_0 + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \cdots\cdots\big)\end{aligned}$$

Comparing coefficient of like powers on both sides, we obtain the unperturbed equation

$$\frac{d^2\Psi_0}{dz^2} + k_1^2\Psi_0 = \lambda_0\Psi_0, \tag{30}$$

The higher powers of ε give us

$$\frac{d^2\Psi_{1m}}{dz^2} + k_2^2(\Psi_{1m} + s(z)\Psi_{0m}) = \lambda_0\Psi_{1m} + \lambda_1\Psi_{0m} \tag{31}$$

and

$$\frac{d^2\Psi_{2m}}{dz^2} + k_2^2(\Psi_{2m} + s(z)\Psi_{1m}) = \lambda_{0m}\Psi_{2m}^{(2)} + \lambda_{1m}\Psi_{1m} + \lambda_{2m}\phi_0, \tag{32}$$

and so on. Multiplying Eq. (31) by taking an inner product with Ψ_{0m} we get

$$\begin{aligned}&\big\langle\Psi_{1m}'', \Psi_{0m}\big\rangle + \big\langle k^2 s(z)\Psi_{0m}, \Psi_{0m}\big\rangle + k^2\big\langle\Psi_{1m}, \Psi_{0m}\big\rangle\\ &\quad = \lambda_0\big\langle\Psi_{1m}, \Psi_{0m}\big\rangle + \lambda_1\big\langle\Psi_{0m}, \Psi_{0m}\big\rangle.\end{aligned} \tag{33}$$

Since the set of eigenfunctions of the depth-equation is complete (Boyles [2]), we may assume the expansion $\Psi_{1m} = \sum_{k=1}^{\infty}\alpha_{km}\Psi_{0k}$.

Using this expansion, properties of the inner product and the orthonormality of the unperturbed eigenfunctions we obtain

$$\lambda_{1m} = \big\langle k^2 s(z)\Psi_{0m}, \Psi_{0m}\big\rangle.$$

Since we are considering the problem of an ocean consisting of two homogeneous layers bounded by a pressure-release surface above and a rigid/reflecting bottom below, we may use the un-perturbed solution for the respective eigenvalues and eigenfunctions presented in Section (2) to obtain

$$\lambda_{1m} = k^2\int_0^h s(z)\Psi_{0m}^2 dz = k^2\int_0^d s(z)\Psi_{0m}^2 dz + k^2\int_d^h s(z)\Psi_{0m}^2 dz.$$

Since $s(z) = 0$ in first layer we get $\lambda_{1m}^{(1)} = 0$ while

$$\lambda_{1m}^{(2)} = k^2\int_d^h s(z)\Psi_{0m}^2 dz. \tag{35}$$

In order to find $\lambda_{2m}^{(2)}$ multiply Eq. (32) by Ψ_{m} and integrate from 0 to h we get

$$\lambda_{2m} = \int_0^d \big(-\lambda_{1m} + k_1^2 s(z)\big)\Psi_{1m}\phi_{0m}^{(1)}\mathrm{d}z + \int_d^h \big(-\lambda_{1\mathrm{m}} + k_2^2 s(z)\big)\Psi_{1\mathrm{m}}\phi_{0\mathrm{m}}^{(2)}\mathrm{d}z$$

or

$$\lambda_{2m}^{(2)} = \int_d^h \Big(-\lambda_{1m}^{(2)} + k_2^2 s(z)\Big)\Psi_{1m}\phi_{0\mathrm{m}}^{(2)}\mathrm{d}z, \tag{36}$$

where $\Psi_{1m} = \sum_{j=1}^{\infty}\alpha_{mj}\Psi_{0m}$.

In order to obtain Ψ_{1m}, we multiply (31) by Ψ_{0n} and integrate from 0 to h to obtain

$$\alpha_{\mathrm{mn}} = \frac{k^2}{\lambda_m - \lambda_n}\int_0^h s(z)\Psi_{0m}\Psi_{0n}\mathrm{d}z,\ m \neq n. \tag{37}$$

It follows that

$$\alpha_{\mathrm{mn}}^{(2)} = \frac{k_2^2}{\lambda_m^{(2)} - \lambda_n^{(2)}}\int_d^h s(z)\phi_{0m}^{(2)}\phi_{0m}^{(2)}\mathrm{d}z,\ m \neq n. \tag{38}$$

Thus the perturbed eigenfunctions are given as a power series in ϵ having the unperturbed ones as the leading entry as follows

$$\Psi_m(z) = \Psi_{0m}(z) + \epsilon\Psi_{1m}(z) + \ldots\ldots\ldots\ldots\ldots$$

where the unperturbed eigenfunctions Ψ_{0m} are given by Eqs. (13) or (20), depending upon the boundary condition at the seabed and

$$\Psi_{1m}(z) = \sum_{j=1}^{\infty}\left(\frac{\mathrm{k}_2^2}{\lambda_{0m}^{(2)} - \lambda_{0j}^{(2)}}\int_d^h s(z)\phi_{0m}^{(2)}\phi_{0j}^{(2)}\mathrm{d}z\right)\phi_j^{(2)}. \tag{39}$$

4. Numerical examples

4.1. Rigid boundary conditions

Example 1. The first example has a linear perturbation on the index of refraction given by $s(z) = (d - z)$. The first and second corrections to the eigenvalues in case of the rigid seabed is obtained from (35) by substituting unperturbed eigenfunctions given by (13) This gives

$$\lambda_{1m}^{(2)} = \frac{k_2^2}{4}\left(-C^2d^2 - C^2h^2 + C^2dh - \frac{C^2}{\gamma_2^2} + \frac{C^2\cos^2\gamma_2(h-d)}{\gamma_2^2}\right) \tag{40}$$

where

$$C = \frac{\sin^2\gamma_2 d}{\cos^2\gamma_2(h-d)}.$$

When $h = d$, we find that

$$\lambda_1^{(2)} = 0$$

which agrees with Duston, Verma and Wood [6] in case of one layer. We also note that if we put $d = 0$, then their solution can be recovered from ours.

Second correction of eigenvalue is obtained numerically as shown in (Table 1):

Table 1

m	$\lambda_{2m}^{(2)}$
1	−0.0002
2	0.1182
3	0.1208
4	0.0962
5	0.1138
6	0.0871
7	0.0221

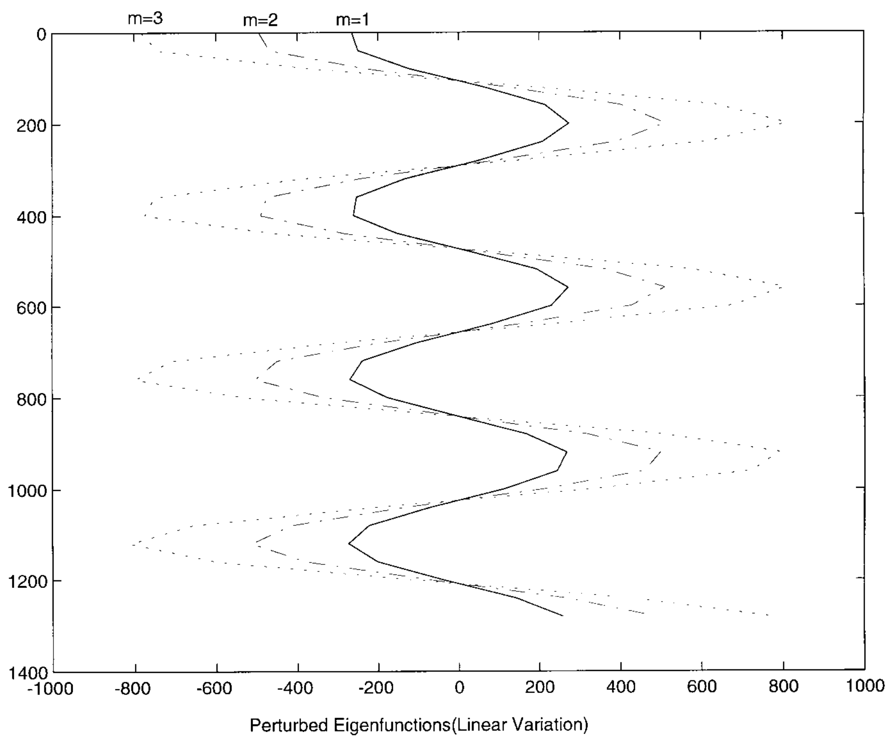

Fig. 2. Linearly perturbed eigenfunctions: Rigid seabed.

Table 2

m	$\lambda_{1m}^{(2)}$	$\lambda_{2m}^{(2)}$
1	0.0102	0.2508
2	0.0077	0.2623
3	0.0062	0.2342
4	0.0050	0.2015
5	0.0029	0.1898
6	0.0029	0.1774
7	0.0022	0.1691

The perturbed eigenfunctions in the case of a rigid seabed are presented in a graph (Fig. 2) for the following values:
$\varepsilon = 0.004$, $h = 1300\,\mathrm{m}$, $d = 750\,\mathrm{m}$, $c_1 = 1500\,\frac{\mathrm{m}}{\mathrm{s}}$, $c_2 = 1495\,\frac{\mathrm{m}}{\mathrm{s}}$ and frequency $= 60\,\mathrm{H}z$.

Example 2. The second example is an idealized model of a symmetrical channel with $s(z) = -2\cos\frac{\pi z}{2d}$.

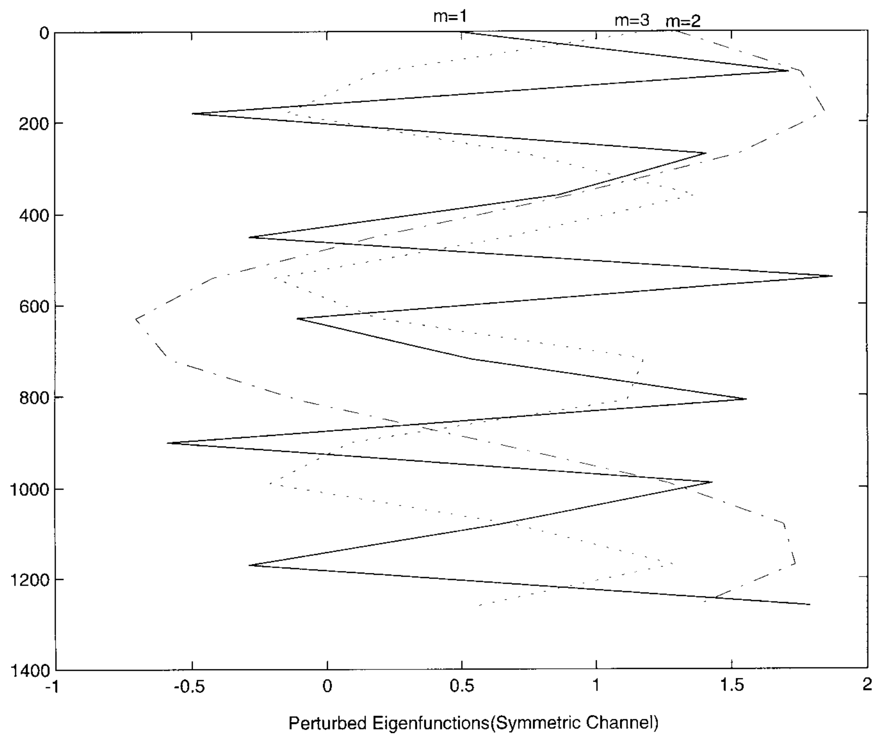

Fig. 3. Symmetric channel: Rigid seabed.

The solution can be written in the form

$$\lambda_1^{(2)} = \frac{(1-T)\left(M\left(2\pi^2 + 16\mathrm{d}^2\gamma^2\right) - \pi N - \left(-\pi^2 + 16d^2\gamma^2\right)\right)}{\pi(\cos^2\gamma_2(-h+d))\left(-\pi^2 + 16\mathrm{d}^2\gamma_2^2\right)}$$

where $M = \sin\frac{1}{2}\pi\frac{h}{d}$, $N = \cos(2\mathrm{d}\gamma_2 - 2\gamma_2 h)$ and $T = \cos^2\gamma_2 d$. When $h = d$, we find that

$$\lambda_1^{(2)} = \frac{(\sin\gamma_2 d)(3\pi - 1)}{\left(-\pi^2 + 16\mathrm{d}^2\gamma_2^2\right)}.$$

First and second corrections are obtained numerically and are shown in (Table 2):

The perturbed eigenfunctions are presented in a graph (Fig. 3) for the following values:

$\varepsilon = 0.004$, $h = 1300\,\mathrm{m}$, $\mathrm{d} = 750\,\mathrm{m}$, $c_1 = 1500\,\frac{\mathrm{m}}{\mathrm{s}}$, $c_2 = 1495\,\frac{\mathrm{m}}{\mathrm{s}}$, $c_2 = 1495\,\frac{\mathrm{m}}{\mathrm{s}}$ and frequency $= 60\,\mathrm{H}z$.

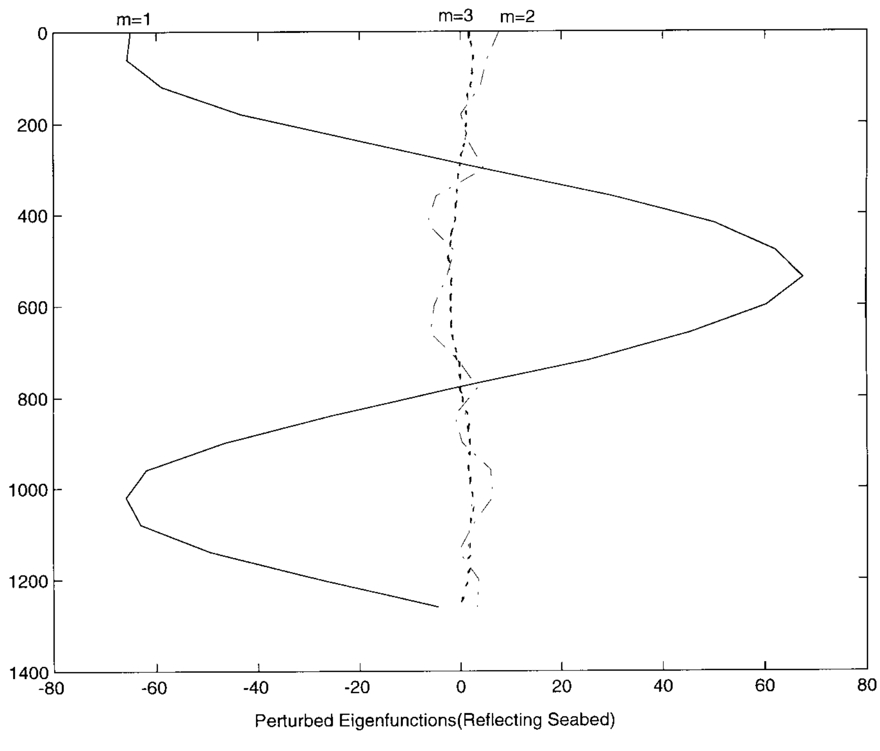

Fig. 4. Perturbed eigenfunctions.

4.2. Impedance boundary conditions

Example 3. Let us consider the linear perturbation on the index of refraction given by

$s(z) = (d - z)$ for the reflecting type seabed satisfying the impedance boundary conditions. We present the first three perturbed eigenfunctions in this case in a graphical form in Fig. (4).

5. Conclusion

We have considered the effects of depth dependent density on the eigenfunctions and eigenvalues of the depth equation resulting from the Helmets equation satisfied by the acoustic pressure in a layered ocean of finite depth. In addition to the more convenient boundary condition assuming the seabed to be rigid, a more general boundary condition, namely the reflecting type seabed condition has been considered. It has been shown that the eigenvalues and eigenfunctions of the depth equation for the rigid seabed can be recovered from those for this general boundary conditions as a special case.

We have used the perturbation method to obtain the eigenvalues and eigenfunctions of the inhomogeneous layered model in which the lower layer is assumed to have depth dependent density and therefore the wave number. In Section (4), we have computed the eigenvalues and plotted eigenfunctions in cases of interest using both the boundary conditions at seabed. The first example presented in Section (4) is that of a linear dependence of the wave number on the depth. In this case the first correction in the eigenvalues is not significant and the eigenmodes remain close to the ones in the unperturbed case. However, in the second example, which is that of a symmetric channel, the effect of the perturbation is more significant. Fig. (3) shows the first three eigen functions in this case. Fig. (4) shows the perturbed eigen functions in the case of a linear perturbation in the wave number. The second and the third mode is seen to become less significant because of the impedance or reflecting boundary condition at the seabed which results in reflection of only a part of the energy at the bottom.

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