Wiener - Hopf Technique

One sided Functions The notion of one sided functions is quite useful in the context of the mixed boundary value problems / half line problems.

We define upper (right) or lower (left) half plane functions, called one sided functions as

$f_+(x) = 0, \quad x < 0$; upper (right) function

$h_-(x) = 0, \quad x > 0$ lower (left) function

Fourier transforms of one sided Functions

As is apparent, the Fourier transform of $f_+(x)$ will be defined over $(0,\infty)$ and that of $f_-(x)$ over $(-\infty, 0)$. Thus

$$\mathcal{F}\{f_+(x)\} = \int_{-\infty}^{\infty} f_+(x)\, e^{i\alpha x}\, dx = \int_0^{\infty} f_+(x)\, e^{i\alpha x}\, dx$$

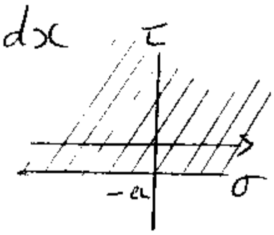
$$= f_+^{*}(\alpha)$$

If $\alpha = \sigma + i\tau$ and $f_+(x) = O\left(e^{-ax}\right)$ as $x \to \infty$ i.e.

$\left| \dfrac{f_+(x)}{e^{-ax}} \right| < C_1$ for large x, a, C_1, constants

Region $\tau > -a$

Then $\left| \int_0^{\infty} f_+(x)\, e^{i(\sigma + i\tau)x}\, dx \right| \le C_1 \int_0^{\infty} \left| e^{-(a+\tau)x} \right| dx$



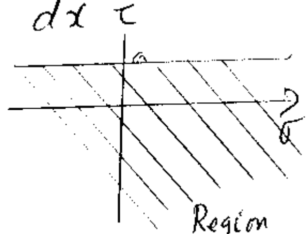
which is absolutely integrable if $\tau > -a$

Thus $f_+^{*}(\alpha)$ is an analytic function of α for $\tau > -a$.

In a similar way,

$$\mathcal{F}\{f_-(x)\} = \int_{-\infty}^{\infty} f_-(x)\, e^{i\alpha x}\, dx = \int_{-\infty}^{0} f_-(x)\, e^{i\alpha x}\, dx$$

$$= f_-^{*}(\alpha)$$



Region $\tau < a$

If $f_-(x) = O(e^{ax})$ as $x \to -\infty$ i.e.

$$\left| \frac{f_-(x)}{e^{ax}} \right| < C_1 \quad \text{for } x \text{ in NHD of } -\infty \ (x \to -\infty)$$

then $\left| \int_{-\infty} f_-(x)\, e^{i(\sigma + i\tau)x}\, dx \right| < C_1 \int_{-\infty}^{0} \left| e^{(a-\tau)x} \right| dx$

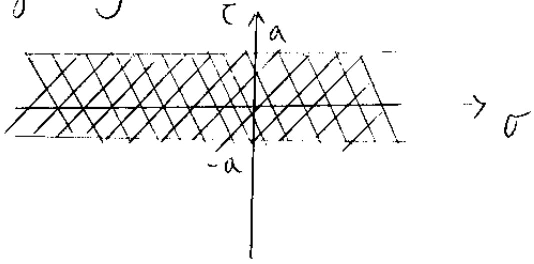
which means $f_-(x)\, e^{i\alpha x}$ is absolutely integrable over $(-\infty, 0)$ if $a - \tau > 0$ or $\tau < a$. Thus $f_-^{*}(\alpha)$ is an analytic function of α for $\tau < a$.

We notice that

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx = \int_{-\infty}^{0} f(x)\, e^{i\alpha x}\, dx + \int_{0}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

so $$f^{*}(\alpha) = f_-^{*}(\alpha) + f_+^{*}(\alpha).$$

With the above discussion, it follows that $f^{*}(\alpha)$ is an analytic function of α for $-a < \tau < a$ provided that $f(x) = O\left(e^{-a|x|}\right)$ as $|x| \to \infty$. The strip of analyticity is shown here.



Integral Equation:

As a motivation, consider the following singular integral equation

$$\int_{-\infty}^{\infty} K(x-\xi)\, u(\xi)\, d\xi = \mu\ u(x) + f(x)\ , \quad -\infty < x < \infty \quad \text{--- (1)}$$

The solution to this integral equation can be readily found taking the Fourier transform and using the convolution theorem to get

$$K^*(\alpha)\ u^*(\alpha) = \mu\, u^*(\alpha) + f^*(\alpha) \qquad \text{--- (2)}$$

so that $\quad u^*(\alpha) = \dfrac{f^*(\alpha)}{K^*(\alpha) - \mu} \qquad \text{--- (3)}$

If the homogeneous equation corresponding to (2) has non-trivial solution, then

$u^*(\alpha)\ [K^*(\alpha) - \mu] = 0$. If $\alpha = \alpha_0$ is a simple zero of $K^*(\alpha) - \mu$ Then $u^*(\alpha) = C\,\delta(\alpha - \alpha_0)$

C some constant.

In the distributional sense $u(x) = C\, e^{-i\alpha_0 x}$ — (4)

However, if the homogeneous equation has no non-trivial solution then (3) can be solved to get a unique solution.

Half-Range Integral Equation:

If the interval is $(0, \infty)$ rather than $(-\infty, \infty)$, The above procedure fails as convolution theorem does not apply.

We consider

$$\int_0^\infty K(x-\xi)\,u(\xi)\,d\xi = \mu\, u(x) + f(x)\,, \quad 0<x<\infty \qquad (5)$$

The so-called Wiener-Hopf Techniques was designed to tackle such half-line problems.

We attack the integral equation (5) by extending the range of integration to $(-\infty,\infty)$ as follows.

$$\int_{-\infty}^{\infty} K(x-\xi)\,u(\xi)\,d\xi = \begin{cases} \mu\, u(x)+f(x), & 0<x<\infty \\ g(x), & -\infty<x<0 \end{cases} \qquad (6)$$

where we now have

$$u(x)=0 \;;\; x<0$$
$$f(x)=0 \;;\; x<0$$
and $$g(x)=0 \;;\; x>0$$

Thus these functions can be referred to as $u_+(x)$, $f_+(x)$ and $g_-(x)$ respectively. In this notation (6) can be written as

$$\int_{-\infty}^{\infty} K(x-\xi)\,u_+(\xi)\,d\xi = \mu\, u_+(x) + f_+(x) + g_-(x) \qquad (7)$$

where $g_-(x)$ is a new un-known function. However, the form of (7) enables us to apply the Fourier transform to obtain

$$k^*(\alpha)\,u_+^*(\alpha) = \mu\, u_+^*(\alpha) + f_+^*(\alpha) + g_-^*(\alpha) \qquad (8)$$

If $K(x) = O(e^{-c|x|})$ as $|x| \to \infty$, $f(x) = O(e^{d'x})$ as $x \to \infty$, where $d' < c$, then the integral on L.H.S of (7) has integrand of order $e^{(d-c)x}$ for $x \to \infty$. Thus for this integral to be analytic, we have $d < c$. In that case, $g_-(x) = O(e^{cx})$ as $x \to -\infty$. We can thus have following:

$K^*(\alpha)$ is analytic for $c < \tau < c$
$u_+^*(\alpha)$ is analytic for $\tau > d$
$f_+^*(\alpha)$ is analytic for $\tau > d'$ and so for $\tau > c$
$g_-^*(\alpha)$ is analytic for $\tau < c$.

The common strip of analyticity for equation (8) is $d < \tau < c$.

Equation (8) is the Wiener-Hopf equation.

Solution of the Wiener-Hopf Eqn.

Re-arrange (8) as

$$u_+^*(\alpha)\{K^*(\alpha) - \mu\} - f_+^*(\alpha) = g_-^*(\alpha) \qquad \text{—— (9)}$$

The term $K^*(\alpha) - \mu$ is mixed (neither + nor −)

We decompose it as (Theorem and example follows)

$$K^*(\alpha) - \mu = m_+^* m_-^*$$

where m_+^* is analytic in the upper half plane while m_-^* is analytic in the lower half plane.

We write (9) as

$$u_+^*(\alpha)\, m_+^* = \frac{f_+^*(\alpha)}{m_-^*} = \frac{g_-^*(\alpha)}{m_-^*} \qquad \text{—(10)}$$

In equation (10) we still have a mixed term $\frac{f_+^*(\alpha)}{m_-^*}$.

We use additive decomposition to write it as

$$\frac{f_+^*(\alpha)}{m_-^*} = P_+^*(\alpha) + P_-^*(\alpha).$$

Equation (10) can then be written as

$$u_+^*(\alpha)\, m_+^* - P_+^*(\alpha) = \frac{g_-^*(\alpha)}{m_-^*} + P_-^*(\alpha) \qquad \text{—(11)}.$$

We notice that the L.H.S of (11) is analytic in the upper half-plane while the R.H.S is analytic in the lower half-plane. For these sides to be equal in the common strip of analyticity, we must have each other as analytic continuation of one another. Thus, these represent an entire function say $E^*(\alpha)$. Now by Liouvilles Theorem, if $|E^*(\alpha)|$ is bounded and entire, then $E^*(\alpha)$ must be a constant. To get this constant, we use estimates on $E^*(\alpha)$ as $|\alpha| \to \infty$. In most cases we can show that $E^*(\alpha) = o(1)$ as $|\alpha| \to \infty$

i.e. $\lim\limits_{|\alpha|\to\infty} \left| \frac{E^*(\alpha)}{1} \right| = 0.$

Thus $E^*(\alpha) = 0$. It then follows that

$$u_+^*(\alpha) = \frac{P_+^*(\alpha)}{m_+^*} \qquad \text{——(12)}$$

The unknown $u_+(x)$ can be obtained using Fourier inversion.

Example: $\int_0^\infty e^{-|x-\xi|} u(\xi)\, d\xi = -\frac{1}{8} u(x) + 1$, $0 < x < \infty$.

Notice that $u(x) = 8 - 8\int_0^\infty e^{-|x-\xi|} u(\xi)\, d\xi$

and so $u(x)$ is bounded as $x \to \infty$. Thus $u_+^*(\alpha)$ is an analytic function of α for $\mathrm{Im}(\alpha) > 0$. In the notation of one sided functions, we write

$$f_+(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Then $f_+^*(\alpha) = \int_0^\infty 1 \cdot e^{i\alpha x}\, dx = \frac{i}{\alpha}$, $\mathrm{Im}(\alpha) > 0$.

The kernel in the integral is $e^{-|x|}$ and

$$\mathcal{F}\{e^{-|x|}\} = \frac{1}{1+\alpha^2} = k^*(\alpha), \quad -1 < \mathrm{Im}\,\alpha < 1$$

$(K(x) = O(e^{-|x|})$ $x \to \infty$. The extension to $(-\infty, \infty)$ will introduce a new unknown function $g_-(x) = \begin{cases} 0, & x < 0, \\ \int_0^\infty e^{-|x-\xi|} u(\xi)\, d\xi & x < 0. \end{cases}$

As indicated in the above procedure, we can write the given integral equation as

$$\int_{-\infty}^\infty e^{-|x-\xi|} u_+(\xi)\, d\xi = -\frac{1}{8} u_+(x) + f_+(x) + g_-(x). \quad \text{—(13)}$$

The Fourier transform of this gives

$$\frac{9+\alpha^2}{8(1+\alpha^2)} u_+^*(\alpha) = \frac{i}{\alpha} + g_-^*(\alpha) \quad \text{—(14)}$$

The Wiener - Hopf equation is written as

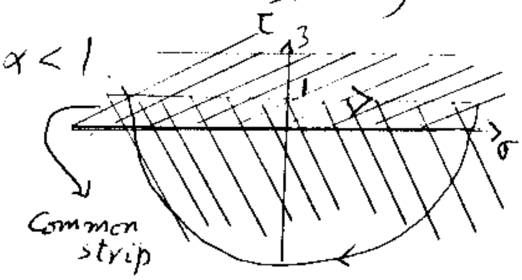
(W8

$$\frac{(\alpha+3i)}{8(\alpha+i)} u_+^*(\alpha) = \frac{i(\alpha-i)}{(\alpha-3i)\alpha} + \frac{\alpha-i}{\alpha-3i} g_-^*(\alpha) \quad \text{——} (15)$$

The term $\dfrac{i(\alpha-1)}{\alpha(\alpha-3i)}$ is still a mixed term. We split it using the partial fractions.

$$\frac{i(\alpha-1)}{\alpha(\alpha-3i)} = P_+(\alpha) + P_-(\alpha) = \frac{i}{3\alpha} + \frac{2i}{3(\alpha-3i)}$$

where $P_+(\alpha) = \dfrac{i}{3\alpha}$ is analytic in $\operatorname{Im}\alpha > 0$ ($\alpha = 0$ lies below this half plane) and $P_-(\alpha) = \dfrac{2i}{3(\alpha-3i)}$ is analytic in $\operatorname{Im}\alpha < 3$ and so $\operatorname{Im}\alpha < 1$.



We can now re-arrange (15) to obtain

$$\frac{(\alpha+3i)}{8(\alpha+i)} u_+^*(\alpha) - \frac{i}{3\alpha} = \frac{2i}{3(\alpha-3i)} + \frac{(\alpha-1)}{(\alpha-3i)} g_-^*(\alpha) \quad \text{—} (1)$$

$$0 < \operatorname{Im}\alpha < 1$$

L.H.S of (19) is analytic in the upper half plane $\operatorname{Im}\alpha > 0$ while the R.H.S is analytic in the lower half plane $\operatorname{Im}\alpha < 1$ and so by Liouville's theorem both sides define an entire function $E^*(\alpha)$. We notice that L.H.S $\to 0$ as $\alpha \to \infty$ and R.H.S $\to 0$ as $\alpha \to -\infty$. Thus $E \equiv 0$. Therefore $E^*(\alpha) \equiv 0$.

We then get from (19), $u_+^*(\alpha) = \dfrac{8i(\alpha+i)}{3\alpha(\alpha+3i)}$ — (17)

Using the inversion integral

$$u(x) = \frac{1}{2\pi}\cdot\frac{8i}{3}\int_{ia-\infty}^{ia+\infty} e^{-i\alpha x}\,\frac{(\alpha+i)}{\alpha(\alpha+3i)}\,d\alpha\,,\quad a>0$$

— (18).

Drawing the contour in the <u>lower half plane</u>, (see shaded zone on (W 8) showing lower-half plane), $\alpha=0$, $\alpha=-3i$ will be the poles inside the contour. If $f(\alpha)$ denotes the integrand only,

$$\operatorname*{Res}_{\alpha=0} f(\alpha) = \lim_{\alpha\to 0}\frac{\cancel{\alpha}\,(\alpha+i)\,e^{-i\alpha x}}{\cancel{\alpha}(\alpha+3i)} = \frac{1}{3}$$

$$\operatorname*{Res}_{\alpha=-3i} f(\alpha) = \lim_{\alpha\to -3i}\frac{\cancel{(\alpha+3i)}(\alpha+i)\,e^{-i\alpha x}}{\alpha\,\cancel{(\alpha+3i)}} = \frac{2}{3}e^{-3x}$$

Thus $u(x) = -\dfrac{4}{3\pi}\,i\,(2\pi i)\left[\dfrac{1}{3}+\dfrac{2}{3}e^{-3x}\right]$

(one minus sign has been introduced as the contour in the lower half plane is clock-wise)

or $u(x) = \dfrac{4}{9}\left[1+2e^{-3x}\right]$.