Solution

DEPARTMENT OF MATHEMATICAL SCIENCES

MATH 301 Methods of Applied Mathematics Term 061

QUIZ #4(a)

Name ______________________ ID # __________ Section # ____

Q1) Find the *Laplace transform* of the following functions

(a) $f(t) = te^{-3t}\sin t$

Consider $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1} = F(s)$

So, by first translation property

$\mathcal{L}\{e^{-3t}\sin t\} = F(s+3) = \dfrac{1}{(s+3)^2+1} = \dfrac{1}{s^2+6s+10} = G(s)$

So, $\mathcal{L}\{t(e^{-3t}\sin t)\} = -\dfrac{dG}{ds} = -\dfrac{-(2s+6)}{(s^2+6s+10)^2} = \dfrac{2s+6}{(s^2+6s+10)^2}$

(4)

(b) $f(t) = e^{2t} * \sin t$. Using $\mathcal{L}\{f*g\} = F(s)\,G(s)$,

$\mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2}$

$\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

So $\mathcal{L}\{e^{2t} * \sin t\} = \dfrac{1}{(s-2)(s^2+1)}$. (3)

Q2. Find inverse Laplace transform $F(s) = \dfrac{1}{s(s^2+9)}$ Using $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{F(s)}{s}$

As $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\} = \dfrac{1}{3}\sin 3t$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+9)}\right\} = \int_0^t \dfrac{1}{3}\sin 3\tau\,d\tau = \dfrac{1}{3}\left(\dfrac{-1}{3}\right)\left[\cos 3\tau\right]_0^t$

(3)

$= \dfrac{1}{9}\left[1-\cos 3t\right].$

Solution

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QUIZ #4(b)

Name ______________________ ID # __________ Section # ____

Q1) Find the *Laplace transform* of the following functions

(a) $f(t) = t\left\{\int_0^t \cos 2\tau d\tau\right\}$.

$$\mathcal{L}\left\{\int_0^t \cos 2\tau \, d\tau\right\} = \frac{\mathcal{L}\{\cos 2t\}}{s} = \frac{s}{s(s^2+4)} = \frac{1}{s^2+4}$$

$$\mathcal{L}\; t\left\{\int_0^t \cos 2\tau \, d\tau\right\} = -\frac{d}{ds}\left\{\frac{1}{s^2+4}\right\} = -\frac{-2s}{(s^2+4)^2}$$

$$= \frac{2s}{(s^2+4)^2} .$$

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(b) $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ t, & t \ge 2 \end{cases}$. We can write $f(t) = 1 + (t-1)\,\mathcal{U}(t-2)$
$= 1 + (t-2)\,\mathcal{U}(t-2) + \mathcal{U}(t-2)$

So $\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{(t-2)\,\mathcal{U}(t-2)\} + \mathcal{L}\{\mathcal{U}(t-2)\}$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s}$$

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Q2. Find inverse Laplace transform $F(s) = \dfrac{1}{s^2(s+1)}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-\tau} d\tau = \left[-e^{-\tau}\right]_0^t = 1 - e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\{s(s+1)\}}\right\} = \int_0^t (1 - e^{-\tau})\, d\tau = \left[\tau + e^{-\tau}\right]_0^t$$

$$= t + e^{-t} - 1$$

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Solution

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QUIZ #4(c)

Name ____________________ ID # __________ Section # ____

Q1. Find inverse Laplace transform $F(s) = \dfrac{1}{s(s^2+1)}$. We use $\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ 3

so, $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\,d\tau = [-\cos\tau]_0^t = 1-\cos t$

Q2) Find the *Laplace transform* of the following functions

(a) $f(t) = e^{2t} * t\sin t$ We use $\mathcal{L}\{f*g\} = F(s)G(s)$

$\mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2}$

Also $\mathcal{L}\{t\sin t\} = -\dfrac{d}{ds}\mathcal{L}\{\sin t\} = -\dfrac{d}{ds}\dfrac{1}{s^2+1} = -\dfrac{-2s}{(s^2+1)^2}$

$= \dfrac{2s}{(s^2+1)^2}$

Thus $\mathcal{L}\{e^{2t} * t\sin t\} = \dfrac{1}{s-2}\cdot\dfrac{2s}{(s^2+1)^2} = \dfrac{2s}{(s-2)(s^2+1)^2}$ 3

(b) $f(t) = \sin t,\ 0 \le t < \pi$
$f(t) = f(t+\pi)$ Periodic with period π.

$\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-\pi s}}\displaystyle\int_0^\pi \sin t\, e^{-st}\,dt$

$= \dfrac{1}{1-e^{-\pi s}}$ Now $I = \displaystyle\int_0^\pi \sin t\, e^{-st}\,dt = \left[\frac{\sin t\, e^{-st}}{-s}\right]_0^\pi + \frac{1}{s}\int_0^\pi \cos t\, e^{-st}\,dt$

$= \dfrac{1}{s}\left[\left[\dfrac{\cos t\, e^{-st}}{-s}\right]_0^\pi - \dfrac{1}{s}\displaystyle\int_0^\pi \sin t\, e^{-st}dt\right]$ $=0$

$I = \dfrac{1}{s}\left[\dfrac{1}{s} + \dfrac{e^{-\pi s}}{s} - \dfrac{1}{s}I\right]$

$\Rightarrow I\left(1+\dfrac{1}{s^2}\right) = \dfrac{1}{s^2} + \dfrac{e^{-\pi s}}{s^2}$

$\Rightarrow I\left(\dfrac{s^2+1}{s^2}\right) = \dfrac{1}{s^2} + \dfrac{e^{-\pi s}}{s^2}$ (4)

$I = \dfrac{1}{s^2+1} + \dfrac{e^{-\pi s}}{s^2+1}$

Thus $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-\pi s}}\left[\dfrac{1}{s^2+1} + \dfrac{e^{-\pi s}}{s^2+1}\right]$