

Introduction to Differential Equations and Linear Algebra

Ordinary Differential Equation (ODE)

An equation that contains one or several (ordinary) derivatives of one or more dependent variables with respect to a single independent variable is called an ODE.

Some ODE's

- (1): $dy/dx=1$; (2): $dy/dx+dw/dx = x$;
 (3): $d^2y/dx^2+dy/dx+y=0$; (4): $d^2y/dx^2+(dy/dx)+y =0$;
 (5): $ydx + (x-1) dy=0$. (6): $dy/dx+x(y)^2=1$ (**nonlin**)
 (7): $(dy/dx)^2=xy$ (**nonlin**) (8): $ydy/dx=1$ (**nonlin**)
 (9): $dy/dx=coty$ (**nonlin**) (10): $y''+p(x)y'+q(x)y=r(x)$
 (**Linear ODE**)

Order of an ODE

The order of the highest derivative in an ODE is called the order of the ODE.

Where & how do the ODE's arise

The ODE's arise in many engineering problems and those situations that represent various natural phenomena:

ODE	Situation represented by the ODE
$dy(t)/dt=k y(t)$ $\int y(t)=c e^{kt}$	Population growth rate (dy/dt) proportional to population present
$d^2y(t)/dt^2=kg$ $\int y(t)=(k/2)gt^2+v_0t+y_0$	Acceleration of a falling object is proportional to "g"
$dT/dt=k(T_{surr}-T_{body})$ $\int dT/dt>0$ if $T_{surr}>T_{body}$ and $dT/dt<0$ if $T_{surr}<T_{body}$	Newton's law of cooling
$dV(t)/dt=-k \sqrt{y}$	Time rate of change of volume proportional to the depth of water

Do All ODE's Have Solutions?

NO

Example 1: $(dy/dx)^2 + y^2 = -1$ has no solution.

Example 2: $(dy/dx)^2 + (y)^2 = 0$ has a *trivial* solution only. ($y=0$ is called *trivial* solution)

Order of ODE's:

Order of the highest derivative that appears in an ODE is called the order of the

ODE.

Example 1: $dy/dx = 1$

1st order ODE

Example 2: $d^2y/dx^2 + dy/dx + y = 0$

2nd order ODE

Example 3: $x^2y''' + 2e^x y'' = (x^2+2)y^2$

3rd order ODE ($y'=dy/dx$)

Representation of ODE's

• $F(x, y, y')=0$,

1st order ODE

• $F(x, y, y', y'')=0$

2nd order ODE

• $F(x, y, y', \dots, y^{(n)})=0$

nth order ODE

Solution of ODE's

$Y = u(x)$ is called a solution of the above ODE's if $F(x, u, u')$, $F(u, u', u'')$, and ... $F(x, u, u', \dots, u^{(n)})$ are all identically equal to zero on some interval (a, b) . This means that all of the ODE's become identities when $y, y', \dots, y^{(n)}$ are replaced respectively by $u, u', \dots, u^{(n)}$.

Initial Value Problem

A problem $dy/dx = f(x, y)$ is called an **initial value** problem if it is subject to a condition that its solution $y(x_0) = y_0$ on an interval containing x_0 . Such a condition gives rise to **unique** solution of the ODE as opposed to a problem with no such condition. By unique means that this condition **determines** the **constant** that appears in the solution of the ODE by a **unique value**.

Example: Find the constant "C" in $y(t) = Ce^{kt}$ that is a solution of the ODE $dy(t)/dt = ky$ subject to $y(0) = 2$.

Solution:

Step 1: $dY(t)/dt = Cke^{kt} = k(Ce^{kt}) = ky$. Thus the given $y(t)$ is a solution of the ODE.

Step 2: $y(0) = C = 2$. Thus the solution of the ODE is $y(t) = 2e^{kt}$.

Integrals as General and Particular Solutions

1st order ODE $dy(x)/dx = f(x)$:

Integrate both sides of the above equation over "x" to get

$$y(x) = \int f(x)dx + C \quad (1)$$

Eq. (1) is called the *General Solution* of the equations $dy/dx = f(x)$ with C integration constant.

Particular Solution of a 1st Order ODE (We understand it by an example)

▮ Solve the initial value problem $dy/dx = 2x+3$ subject to $y(1) = 2$.

Solution: Integrate both sides of the above equation to get:

$$Y(x) = x^2 + 3x + C$$

The above solution is a general solution. Now we substitute the initial condition $y(1)=2$ to find that:

$$2 = (1)^2 + 3(1) + c \implies c = -2.$$

This condition gives particular solution of the above equation:

$$Y(x) = x^2 + 3x - 2$$

Velocity and Acceleration:

□ The **motion (curve)** of a particle along a straight line is represented by $x = f(t)$.

□ At any time " t " the velocity of the particle is defined by: $\underline{V} = dx/dt = f'(t)$.

□ The acceleration of the particle is represented by:

$$\underline{a} = d\underline{V}/dt = d^2x/dt^2$$

□ Newton's second law implies from here: $\underline{F} = ma \implies \underline{F} = m$

$$d^2x/dt^2$$

The above equation being a **2nd** order ODE there are **two constants** involved and are determined by **initial position** $X(0)=X_0$ and **initial velocity** $V(0)=V_0$ in the following way:

Write $dV/dt = a$ (a being constant) and integrate with respect to " t " to get

$$V = dx/dt = \int a dt + A = at + A. \quad (1)$$

- Require that $V(0)=V_0$ to get $V_0=A$. Thus equation (1) becomes:

$$V = dx/dt = at + V_0 \quad (2)$$

- Integrate **equation (2)** over " t " once again to get:

$$x(t) = \frac{1}{2}at^2 + V_0t + B \quad (3)$$

- Require that $X(0)=X_0$ to get $B=X_0$. Thus equation (3) takes the form:

$$x(t) = \frac{1}{2}at^2 + V_0t + X_0 \quad (4)$$

1.4: Separable Equations and their Applications

1st order Separable ODE $dy(x)/dx = F(x, y)$:

The above equation is called separable if we can write: $F(x, y) = f(x)/g(y)$. Thus the ODE becomes:

$$dy/dx = f(x)/g(y) \Rightarrow g(y)dy = f(x)dx$$

(1)

Integrating (1) on both sides we get:

$$H(y) = K(x) + C, \text{ where}$$

$$H(y) = \int g(y)dy \text{ and } K(x) = \int f(x)dx$$

(2)

Illustration by Example: Solve $dy/dx = -6xy$ subject to $y(0)=7$.

Solution:

▮ The ODE is separable and becomes: $dy/y = -6x dx$.

$$\ln y = -3x^2 + \ln C \text{ or}$$

▮ Its general solution is: $y = Ce^{-3x^2}$

▮ Initial condition gives: $7 = C$

▮ Thus the initial value solution becomes: $y = 7e^{-3x^2}$

Example: Find general solution of the $dy/dx + 2xy^2 = 0$.

Rewrite the differential equations as

$$dy/dx = -2xy^2$$

Separate variables

$$\frac{dy}{y^2} = -2x dx$$

Integrate both sides

$$\int \frac{dy}{y^2} = \int -2x dx$$

which gives

$$y^{-1} = (-x^2 + c)$$

$$\text{or } y = 1/(c - x^2)$$

First Order Linear Differential Equation

Consider the ODE $dy/dx + P(x)y = Q(x)$ where $P(x)$ & $Q(x)$ are continuous on the interval of solution of the ODE. Then the **integrating factor**, $\rho(x)$, is defined by:

$$\rho(x) = e^{\int P(x)dx}$$

Multiply by this integrating factor on both sides and write the result as:

$$\frac{d}{dx}(e^{\int P(x)dx} y) = Q(x)e^{\int P(x)dx}$$

Integrating the above equation on both sides gives:

$$y(x) = e^{-\int P(x)dx} [\int Q(x)e^{\int P(x)dx} dx + c]$$

▮ **Example:** Solve the ODE $\frac{dy}{dx} - y = \frac{11}{8}e^{-\frac{x}{3}}$ subject to $y(0) = -1$.

▮ **Solution.**

Step 1. The IF for the above ODE is: $\rho(x) = e^{\int -dx} = e^{-x}$

Step 2. The ODE becomes: $\frac{d}{dx}(e^{-x}y) = \frac{11}{8}e^{-\frac{4x}{3}}$

Step 3. Integrating above gives: $y(x) = -\frac{33}{32}e^{-\frac{x}{3}} + e^xc$

Step 4. Use the initial condition $y(0) = -1$ to get:

$$-1 = -\frac{33}{32} + c \Rightarrow c = \frac{1}{32}$$

Step 5. The solution of the ODE becomes:

$$y(x) = (-33e^{-\frac{x}{3}} + e^x)/32$$

