Non Linear Differential Equations

Finding solutions to non-linear differential equations is a challenging task and often exact solutions can not be found. There is another important difference: The singularities of solution to a linear DE are inherited from the coefficients of DE (ref. singular points). However, a non-linear DE may have singularities arising "spontaneously" (moving singularities). To highlight the difference we consider following:

Example (1) (Linear D.E.)

Consider $\frac{dy}{dx} + \frac{y}{x-1} = 0$, $y(0) = 1$

It has a regular singular point at $x = 1$. It is straightforward to see that the solution $y = \frac{1}{1-x}$ has a pole at $x = 1$.

Change of initial condition (to say $y(0) = A$) changes y to $y = \frac{A}{1-x}$ and no change in the position of pole takes place.

Example (2) (Non-linear D.E.)

Consider $y' = \frac{y^2}{1-xy}$, $y(0) = 1$.

One can find a series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad a_0 = 1 \text{ (due to initial conditions)}$$

We can find $a_n = \frac{(n+1)^{n-1}}{n!}$

Applying ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e}$

Hence the series is valid (convergent) for $|x| < \frac{1}{e}$. Thus we find $x = \frac{1}{e}$ may be a singularity of the solution. In order to further see this singularity, let us use iteration

$$y_{n+1} = e^{x y_n}, \quad y_0 = y(0) = 1$$

then

$$y_1 = e^{x}$$

$$y_2 = e^{x e^{x}}$$

$$y_3 = e^{x e^{x e^{x}}} \quad \cdots$$

Then $y(x) = \lim_{n\to\infty} y_n(x)$

The sequences increases without bounds if $x > \frac{1}{e}$

$\left[\; y_1 > y(0), \quad y_2 > y_1, \quad \ldots, \quad y_{n+1} > y_n \text{ and so on} \;\right]$

Hence we find that $x = \frac{1}{e}$ is a singularity which could not be anticipated from the coefficients. This is a <u>spontaneous singularity</u>.

<u>Example</u> (First Order DE) Riccati equation

Consider $y' = y^2 + x$

Let us use $y(x) = x^{1/2} u(x)$

$$y' = \frac{1}{2} x^{-1/2} u(x) + x^{1/2} u'(x)$$

$$DE \Rightarrow x^{1/2} u' + \frac{1}{2} x^{-1/2} u(x) = x u^2 + x$$

Re-arranging

$$u' = (1+u^2)\, x^{1/2} - \frac{u}{2x}$$

The last term can be neglected in the favr of first on R.H.S as

$$x^{-1} << x^{1/2} \quad , x \to \infty$$

$$\text{and} \quad u << 1+u^2$$

Thus $\quad u' \sim (1+u^2)\, x^{1/2} \quad , x \to \infty$

Solving as D.E

$$\frac{du}{dx} = (1+u^2)\, x^{1/2}$$

$$\frac{du}{1+u^2} = x^{1/2}\, dx$$

gives $\quad u = \tan\left(\frac{2}{3} x^{3/2} + C(x)\right)$

$$C(x) << x^{3/2}, x \to \infty$$

Thus $\quad u \sim \tan[\phi(x)]$

where $\phi(x) \sim \frac{2}{3} x^{3/2} \quad , x \to \infty$

This gives $\quad y(x) \sim x^{1/2} \tan[\phi(x)]$

where $\phi(x) \sim \frac{2}{3} x^{3/2} \quad , x \to \infty$

Due to the presence of tangent term, one expects an infinite number of spontaneous singularities

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We re-call that all solutions of nonlinear D.E's may not be found from the general solution. For example $y' = x y^{1/2}$ has general solution $y = (x^2/4 + c)^2$. However, the trivial solution $y=0$ can not be obtained by choosing any value of parameter c. This characteristic of nonlinear D.E's will be there in asymptotic solutions also.

Higher Order D.E's

Example (1) $\quad y^3 y'' = -1, \quad x \to \infty$

Find complete description of the asymptotic behavior of $y(x)$.

We may observe that the solution can not be expected to have an exponential behavior as exponential term would not cancel (eqn is not equi dimensional). To check for algebraic behavior, we can test

$$y(x) \sim A x^{\alpha}, \quad x \to \infty$$

$$y' = A\alpha x^{\alpha-1}, \quad y'' = A\alpha(\alpha-1) x^{\alpha-2}$$

$$\text{D.E} \Rightarrow A^4 \alpha(\alpha-1) x^{4\alpha-2} \sim -1, \quad x \to \infty.$$

This gives $\alpha = \frac{1}{2}$, $A = \pm\sqrt{2}$.

There is no arbitrary constant in resulting asymptotic solution

$$y(x) \sim \pm\sqrt{2x}, \quad x \to \infty$$

However, we notice $x \to x + a$ leaves the equation invariant (translation invariant). Thus we can say

$$y(x) \sim \pm\sqrt{2(x+a)}, \quad x \to \infty.$$

The solution is not most general solution as we could expect two arbitrary constants.

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Due to second derivative in D.E. we could have a linear term in x which vanishes on twice differentiation. This is not included in the above solution. We thus assume

$$y(x) \sim bx + c + \varepsilon(x), \quad x \to \infty$$

where $\varepsilon(x) \to 0$ as $x \to \infty$.

D.E $\Rightarrow$ $\varepsilon''(x)\left[b^3x^3 + \varepsilon^3 + 3bc^2x + 3b^2x^2c\right] \sim -1, \; x \to \infty$

Retaining dominating terms only

$$b^3x^3\varepsilon''(x) \sim -1, \quad x \to \infty$$

so that $\varepsilon(x) \sim -\left[2b^3x\right]^{-1}, \; x \to \infty.$

Note that it satisfies requirement $\varepsilon \to 0, \; x \to \infty$.

Thus a better representation will be

$$y(x) \sim bx + c + \frac{d}{x} + \frac{e}{x^2}, \; \cdots, \quad x \to \infty$$

Putting in D.E

$$\left(\frac{2d}{x^3} + \frac{6e}{x^4} + \cdots\right)\left(b^3x^3 + 3b^2cx^2 + \cdots\right) \sim -1 \quad x \to \infty$$

$$\sim \quad 2db^3 + 6(b^3e + b^2cd)/x + \cdots \quad \sim -1, \quad x \to \infty.$$

Comparing like terms

$$2db^3 = -1 \quad \Rightarrow d = -\frac{1}{2b^3}$$

$$6(b^3e + b^2cd) = 0 \quad \Rightarrow \quad e = \frac{c}{2b^4}, \cdots$$

Thus $y(x) \sim bx + c - \dfrac{1}{2b^3x} + \dfrac{c}{2b^4x^4} \cdots, \; x \to \infty$

Computing more terms, we can see

$$y(x) = \pm\left[\frac{(c_1x + c_2)^2 - 1}{c_1}\right]^{1/2}, \; c_1 \neq 0$$

(Expand later as Binomial Series!)

Painleve transcendents.

Example Consider $y'' = y^2 + x$, $x \to \infty$

[One of the set of six equations whose solutions are Painleve transcendents)

From the equation as x is large, $y'' > 0$. Thus curvature of $y(x)$ is positive so the solution may become ~~unbounded~~ (singular) at a finite value of y)

Let us try $$y(x) \sim \frac{A}{(x-a)^b}, \quad x \to a$$

$$y'(x) \sim -\frac{Ab}{(x-a)^{b+1}}, \quad x \to a$$

$$y''(x) \sim \frac{A(b)(b+1)}{(x-a)^{b-2}}$$

$$\text{DE} \Rightarrow \frac{A\,b(b+1)}{(x-a)^{b+2}} \sim \frac{A^2}{(x-a)^{2b}} + x$$

The asymptotic result suggests by comparing powers of $(x-a)$ that

$$b = 2 \text{ and } A = 6.$$

Hence $$y(x) \sim \frac{6}{(x-a)^2}, \quad x \to \infty$$

The conclusion from this step is that $y(x)$ has (spontaneous) movable second order poles. To see if these second order poles can be studied further, we use the same transformation as in the previous example

$$y = \sqrt{x}\ u(x)$$

$$DE \Rightarrow \quad u'' = \sqrt{x}(u^2+1) - \frac{u'}{x} + \frac{u}{4x^2}$$

In order to get rid of $\sqrt{x}$ as coefficient of u^2+1, we put $x = s^{4/5}$ (change of independent variable)

This gives

$$\frac{d^2u}{ds^2} = \frac{16}{25}(u^2+1) - \frac{1}{s}\frac{du}{ds} + \frac{4}{25}\frac{u}{s^2}$$

terms containing $\frac{1}{s}$ and $\frac{1}{s^2}$ can be neglected for large s (which is the case as $x \to \infty$)

Thus we get

$$\frac{d^2u}{ds^2} \sim \frac{16}{25}(u^2+1) \quad , s \to \infty$$

Multiply by u' and integrate

$$\frac{1}{2}\left(\frac{du}{ds}\right)^2 \sim \frac{16}{25}\left(\frac{u^3}{3} + u + C\right) \quad , s \to \infty$$

or $$\left(\frac{du}{ds}\right) \sim \pm\frac{4}{5}\left(\frac{u^3}{3} + 2u + 2C\right)^{-1/2} \quad , s \to \infty$$

or $$\int \frac{du}{2\left(\frac{u^3}{3}+2u+2C\right)} \sim \pm\frac{4}{5}\int ds = \pm\frac{4}{5}s \quad , s \to \infty$$

The answer is in terms of what are known as <u>elliptic functions</u>. By properties of elliptic functions $u(s)$ has infinite many poles separated by period P.