# Chapter 8
## Laplace Transforms

<u>Defn</u>: The Laplace transform of a function $f(t)$, $t>0$ is defined to be $F(s)$ given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad , \text{ if it exists.}$$

If $f(t)$ is of exponential order, i.e. $|f(t)| \leq K e^{at}$ for $K$, $a$ real constants then we find

$$\left|\int_0^\infty f(t)\, e^{-st} dt\right| \leq \left|\int_0^\infty K e^{at} e^{-st} dt\right| \leq |K| \left|\int_0^\infty e^{-(s-a)t} dt\right|$$

$$= |K| \left[\frac{e^{-(s-a)}}{s-a}\right]_0^\infty$$

The limit as $t \to \infty$ exists if $s-a>0$ or $s>a$.

Thus $F(s)$ for such a function exists for $s>a$.

<u>Examples</u> ① $\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$, $s>0$

② $\mathcal{L}\{t\} = \int_0^\infty \underbrace{t}_{u}\, \underbrace{e^{-st} dt}_{dv} = \cancel{\left[\frac{t\, e^{-st}}{-s}\right]_0^\infty}^{=0} + \int_0^\infty \frac{e^{-st}}{s} dt = \frac{1}{s^2}$, $s>0$.

③ $\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st} dt = \left[\frac{t^2 e^{-st}}{-s}\right]_0^\infty + \frac{2}{s}\int_0^\infty t\, e^{-st} dt = \frac{2}{s}\cdot\frac{1}{s^2}$

$$= \frac{2}{s^3}; \quad s>0.$$

④ $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ (Following above integration by parts $n$ times)

⑤ $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{+at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$

$$= \frac{1}{s-a}, \quad s>a.$$

Linearity Property : If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$.

In fact, $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \int_0^\infty \{\alpha f(t) + \beta g(t)\} e^{-st}\, dt$

$$= \alpha \int_0^\infty f(t) e^{-st} dt + \beta \int_0^\infty g(t) e^{-st} dt = \alpha F(s) + \beta G(s).$$

First Shifting Property : (Shifting on $s$ - axis)

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(s-a)t} dt$$

Compare with $F(s) = \int_0^\infty f(t) e^{-st} dt$,

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

More Examples

(6) Find (a) $\mathcal{L}\{\sin \omega t\}$ (b) $\mathcal{L}\{\cos \omega t\}$.

We solve (a) and (b) by two different methods. However both can be solved by either.

(a) $\mathcal{L}\{\sin \omega t\} = F(s) = \int_0^\infty \sin \omega t \, e^{-st} dt$

Integration by parts gives,

$$F(s) = \left[\cancel{\frac{\sin \omega t \, e^{-st}}{-s}}^{=0}\right]_0^\infty + \frac{\omega}{s} \int_0^\infty \cos \omega t \, e^{-st} dt$$

$$= \frac{\omega}{s}\left[\left\{\frac{\cos \omega t \, e^{-st}}{-s}\right\}_0^{\infty 0} - \frac{\omega}{s}\int_0^\infty \sin \omega t \, e^{-st} dt\right]$$

or

$$F(s) = \frac{\omega}{s^2} - \frac{\omega^2}{s^2} F(s) \implies \left(1 + \frac{\omega^2}{s^2}\right) F(s) = \frac{\omega}{s^2}$$

$$\left(\frac{s^2+\omega^2}{s^2}\right) F(s) = \frac{\omega}{s^2}$$

$$\implies \boxed{F(s) = \frac{\omega}{s^2+\omega^2}}$$

(b) We can write

$$\cos\omega t = \frac{1}{2}\left(e^{+i\omega t} + e^{-i\omega t}\right)$$

So, $\mathcal{L}\{\cos\omega t\} = \mathcal{L}\left\{\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\right\}$

By linearity property,

$$\mathcal{L}\{\cos\omega t\} = \frac{1}{2}\left\{\mathcal{L}(e^{i\omega t}) + \mathcal{L}(e^{-i\omega t})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right\} = \frac{1}{2}\left\{\frac{s+i\omega+s-i\omega}{s^2+\omega^2}\right\}$$

$$= \frac{1}{2}\,\frac{2s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}.$$

(7) (a) $\mathcal{L}\{e^{at}\cos\omega t\}$, (b) $\mathcal{L}\{e^{at}\sin\omega t\}$ (c) $\{\mathcal{L}\{te^{at}\}$

(a) Using $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ we get from (6b)

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}.$$

Similarly (b) and (c).

<u>Inverse Laplace transform</u>:

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\} = f(t)$, $t>0$.

For example $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t.$$

## A useful Result :

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{t\, f(t)\} = - F'(s).$$

For $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$.

Differentiate w.r.t. $s$

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st} dt = \int_0^\infty \frac{d}{ds}\{f(t) e^{-st} dt\}$$

$$= -\int_0^\infty \underbrace{t\, f(t)}\, e^{-st}\, dt.$$

Thus $\mathcal{L}\{t\, f(t)\} = - F'(s)$.

Note we can extend this result to,

$\mathcal{L}\{t^2 f(t)\} = (-1)^2 F''(s)$, and in general

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

## Examples

(8) $\mathcal{L}\{t \cos \omega t\}$

As $\mathcal{L}\{\cos \omega t\} = F(s) = \dfrac{s}{s^2+\omega^2}$

$$\mathcal{L}\{t \cos \omega t\} = - F'(s) = -\frac{d}{ds}\left[\frac{s}{s^2+\omega^2}\right]$$

$$= - \frac{s^2+\omega^2-2s^2}{(s^2+\omega^2)^2} = -\frac{\omega^2-s^2}{(s^2+\omega^2)^2} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

(9) Find Laplace inverse transform

(a) $\dfrac{1}{s^2+9}$ (b) $\dfrac{s-1}{(s-1)^2+4}$

(c) $\dfrac{1}{(s^2+1)^2}$ (d) $\cancel{\dfrac{1}{s^4+1}}$ $\dfrac{1}{s^4-1}$

(a) As $\mathcal{L}\{\sin 3t\} = \dfrac{3}{s^2+9}$ so

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3} \cdot \sin 3t.$$

(b) As $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \cos 2t$ so by first shifting property

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} = e^t \cos 2t$$

(c) If $f(t) = \sin t$, Then $F(s) = \dfrac{1}{s^2+1}$

$$F'(s) = \frac{-1}{(s^2+1)^2}$$

Hence $\mathcal{L}\{t \sin t\} = -F'(s) = \dfrac{1}{(s^2+1)^2}$.

i.e. $\mathcal{L}^{-1}\dfrac{1}{(s^2+1)^2} = t \sin t$.

(d) $\dfrac{1}{s^4-1} \equiv \dfrac{1}{(s^2+1)(s+1)(s-1)} \equiv \dfrac{A}{s+1} + \dfrac{B}{s-1} + \dfrac{Cs+D}{s^2+1}$

Multiply by $s^4-1$

$$1 \equiv A(s-1)(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s+1)(s-1)$$

$s=1 \Rightarrow 4B=1 \Longrightarrow B = \dfrac{1}{4}$

$s=-1 \Rightarrow \quad 1 = -4A \Rightarrow A = -\frac{1}{4}$

Compare Coefficients of $s^3$

$$0 = A + B + C \Rightarrow C = -A - B = \frac{1}{4} - \frac{1}{4} = 0$$

Compare coefficients of $s^2$

$$0 = -A + B + D \Rightarrow D = A - B = -\frac{1}{4} - \frac{1}{4}$$
$$= -\frac{1}{2}$$

Hence the partial fractions of $\frac{1}{s^4 - 1}$ are

$$\frac{1}{s^4-1} \equiv -\frac{1}{4}\,\frac{1}{s+1} + \frac{1}{4}\,\frac{1}{s-1} - \frac{1}{2}\,\frac{1}{s^2+1}$$

Therefore $\mathcal{L}^{-1}\left\{\frac{1}{s^4-1}\right\} = -\frac{1}{4}e^{-t} + \frac{1}{4}e^{t} - \frac{1}{2}\sin t.$

<u>Laplace transform of Derivatives</u>. Assume $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{f'(t)}_{dv}\,\underbrace{e^{-st}}_{u}\,dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt$$

$$= -f(0) + s\,F(s)$$

Hence $\mathcal{L}\{f'(t)\} = s\,F(s) - f(0).$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty \underbrace{f''(t)}_{dv}\,\underbrace{e^{-st}}_{u}dt = \left[f'(t)e^{-st}\right]_0^\infty + s\int_0^\infty f'(t)e^{-st}dt$$

$$= -f'(0) + s\,\mathcal{L}\{f'(t)\} = -f'(0) + s^2F(s) - sf(0)$$

so that

$$\mathcal{L}\{f''(t)\} = s^2F(s) - s\,f(0) - f'(0).$$

Laplace transform of higher derivatives can similarly be worked out.

# Application to Solution of ODE's

Example: $y'' + 4y = 3\cos 2t$, $y(0) = 1$, $y'(0) = 0$.

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0)$$
$$= s^2 Y(s) - s$$
$$\mathcal{L}\{4y\} = 4\,Y(s)$$
$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

Hence ODE $\Rightarrow$

$$s^2 Y(s) - s + 4Y(s) = \frac{3s}{s^2+4}$$

or $Y(s)\{s^2+4\} = s + 3\dfrac{s}{s^2+4}$

or $Y(s) = \dfrac{s}{s^2+4} + 3\dfrac{s}{(s^2+4)^2}$ ——— $(*)$

Now $F(s) = \dfrac{s}{s^2+4} \Rightarrow \mathcal{L}^{-1}(F(s)) = \cos 2t$.

$F'(s) = \cancel{\dfrac{s^2+4 - \ldots}{(s^2+4)^2}}$

To find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\}$ we use property

$\mathcal{L}\{t\,f(t)\} = -F'(s)$.

If $F(s) = \dfrac{1}{s^2+4}$, $f(t) = \dfrac{1}{2}\sin 2t$

$F'(s) = \dfrac{-2s}{(s^2+4)^2}$ or $-F'(s) = 2\cdot\dfrac{s}{(s^2+4)^2}$

Hence $\mathcal{L}^{-1}\Big\{\underbrace{2\cdot\dfrac{s}{(s^2+4)^2}}_{-F'(s)}\Big\} = \{t\,f(t)\} = t\cdot\dfrac{1}{2}\sin 2t$

Thus $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}\, t \sin 2t$.

Using these results, (*) gives

$$y(t) = \cos 2t + \frac{3}{4}\, t \sin 2t$$

We have used $\mathcal{L}\{t f(t)\} = -F'(s)$ above.

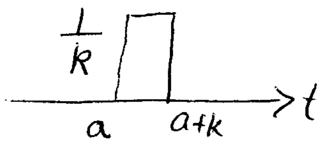
## Dirac Delta Function:

Many practical situations in mechanics, signal processing or geophysical problems are modelled by the so called Dirac delta function,

$$\delta(t-a) = 0, \quad t \neq a, \quad \int_a^b \delta(t-a)\,dt = 1.$$

In order to find Laplace transform, let us consider

$$f_K(t) = \begin{cases} \frac{1}{k}, & a < x < a+k \\ 0, & \text{otherwise} \end{cases}$$




Then $\lim_{K\to 0}\{f_K(t)\} = \delta(t-a)$.

Now $\mathcal{L}\{f_K(t)\} = \int_0^\infty f_K(t)\, e^{-st}\,dt = \int_a^{a+k} \frac{1}{k}\, e^{-st}\,dt$

$$= \frac{1}{k}\left[\frac{e^{-st}}{-s}\right]_a^{a+k} = \frac{-1}{k}\left[\frac{e^{-s(a+k)} - e^{-sa}}{s}\right] = \frac{e^{-as}}{s}\left[\frac{1-e^{-sk}}{k}\right]$$

If we now take limit as $k \to 0$ and assume that

$\lim_{k\to 0} \mathcal{L}\{f_K(t)\} = \mathcal{L}\{\lim_{k\to 0} f_K(t)\}$, then

$$\mathcal{L}\{\delta(t-a)\} = \frac{e^{-as}}{s} \lim_{k\to 0}\left[\frac{1-e^{-sk}}{k}\right].$$

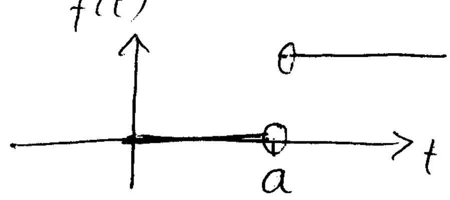
By L'Hopital rule,

$$\mathcal{L}\{\delta(t-a)\} = \frac{e^{-as}}{s} \lim_{k\to 0} \frac{s\,e^{-sk}}{1} = \frac{e^{-as}}{s}\cdot s = e^{-as}$$

Thus $\mathcal{L}\{\delta(t-a)\} = e^{-as}$.

Unit Step Function : We define

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$




$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)\, e^{-st}\, dt = \int_a^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$

Second Shifting Property (Shifting on t - axis)

$$\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as} F(s).$$

$$\mathcal{L}\{f(t-a)\, u(t-a)\} = \int_0^\infty f(t-a)\, u(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty f(t-a)\, e^{-st}\, dt \qquad \text{as } u(t-a)=0,\ t<a.$$

Put $u = t-a$, $du = dt$

$t=a \Rightarrow u=0$, $t=\infty \rightarrow u=\infty$

This gives

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = \int_0^\infty f(u)\, e^{-s(u+a)}\, du$$

$$= e^{-as}\int_0^\infty f(u)\, e^{-su}\, du = e^{-as} F(s).$$

This shows the required Result.

<u>Another form</u>: We can write this result as

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\,\mathcal{U}(t-a).$$

<u>Example</u>: Solve the differential equation

$$y'' + y = \delta(t-2\pi)$$

$$y(0) = 0, \quad y'(0) = 1.$$

<u>Solution</u> $\mathcal{L}(y'') = s^2 Y(s) - s\,y(0) - y'(0)$

$$= s^2 Y(s) - 1$$

$$\mathcal{L}\{\delta(t-2\pi)\} = e^{-2\pi s}$$

D.E $\Rightarrow$ $s^2 Y(s) - 1 + Y(s) = e^{-2\pi s}$

or $(s^2+1)\,Y(s) = 1 + e^{-2\pi s}$

or $Y(s) = \dfrac{1}{s^2+1} + \dfrac{e^{-2\pi s}}{s^2+1}$

As $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$

$y(t) = \sin t + \sin(t-2\pi)\,\mathcal{U}(t-2\pi)$

or $y(t) = \sin t + \sin t\,\mathcal{U}(t-2\pi)$.