Chapter 9

Fourier Transforms and Applications To PDE's

Fourier transform of a function $f(x)$, $-\infty < x < \infty$, is defined as

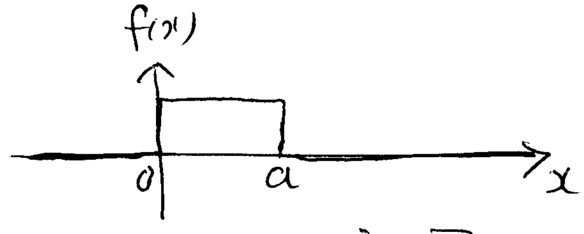
$$\mathcal{F}\{f(x)\} = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx ,$$

if the integral exists.

The inversion is defined as

$$\mathcal{F}^{-1}\{\hat{f}(\alpha)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha$$

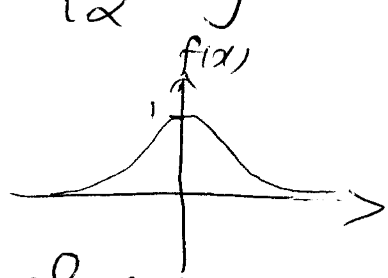
Examples: ① Consider $f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$



$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{a} e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-i\alpha x}}{i\alpha} \right]_0^a = \frac{1}{\sqrt{2\pi}} \left[\frac{1 - e^{-i\alpha a}}{i\alpha} \right].$$

② $f(x) = e^{-|x|}$



$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|}\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{(1-i\alpha)x}\, dx + \int_{0}^{\infty} e^{-(1+i\alpha)x} \right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[\frac{e^{(1-i\alpha)x}}{(1-i\alpha)}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+i\alpha)x}}{-(1+i\alpha)}\right]_{0}^{\infty}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{1-i\alpha} + \frac{1}{1+i\alpha}\right\}$$

$$\frac{1}{\sqrt{2\pi}}\left\{\frac{1+\cancel{i\alpha}+1-\cancel{i\alpha}}{1+\alpha^2}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\alpha^2}$$

③ $f(x) = e^{-ax^2}, \quad a > 0$

$$\mathcal{F}\{f(x)\} = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left[-ax^2 - i\alpha x\right] dx$$

Write $\left[-ax^2 - i\alpha x\right] = -\left(\sqrt{a}\,x + \frac{i\alpha}{2\sqrt{a}}\right)^2 + \left(\frac{i\alpha}{2\sqrt{a}}\right)^2$

so that $\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{a}\,x + \frac{i\alpha}{2\sqrt{a}}\right)^2 + \left(\frac{i\alpha}{2\sqrt{a}}\right)^2\right] dx$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\alpha^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{a}\,x + \frac{i\alpha}{2\sqrt{a}}\right)^2\right] dx$$

Putting $I = \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{a}\,x + \frac{i\alpha}{2\sqrt{a}}\right)^2\right] dx$

and using $u = \sqrt{a}\,x + \frac{i\alpha}{2\sqrt{a}}$, $du = \sqrt{a}\,dx$,

we get $I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2}\, dv = \frac{1}{\sqrt{a}}\sqrt{\pi} = \sqrt{\frac{\pi}{a}}$

Hence $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\alpha^2}{4a}} \cdot \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}}\, e^{-\frac{\alpha^2}{4a}}.$

Linearity Property

$$\mathcal{F}(af+bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g)$$

Shifting Property.

$$\mathcal{F}\{f(x-a)\} = e^{-i\alpha a}\,\mathcal{F}\{f(x)\}.$$

For, $\mathcal{F}\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{-i\alpha x}\,dx$

Put $u = x-a$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha(u+a)}\,du = \frac{e^{-i\alpha a}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\,du$$

which gives the required result.

Fourier transform of Derivatives

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{-i\alpha x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left[f(x)\, e^{-i\alpha x}\right]_{-\infty}^{\infty} + i\alpha\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\,dx\right]$$

$$= i\alpha\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\,dx = i\alpha\,\hat{f}(\alpha).$$

$$\mathcal{F}\, f''(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f''(x)\, e^{-i\alpha x}\,dx$$

$$= (i\alpha)^2\,\hat{f}(\alpha) = -\alpha^2\,\hat{f}(\alpha).$$

Example: Find $\mathcal{F}\{x e^{-x^2}\}$.

Notice that $\left(e^{-x^2}\right)' = -2x e^{-x^2}$

Hence $\mathcal{F}\{x e^{-x^2}\} = \mathcal{F}\left\{-\frac{1}{2}\left(e^{-x^2}\right)'\right\}$

$$= -\frac{1}{2}\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\}$$

$$= -\frac{1}{2}(i\alpha)\,\mathcal{F}\{e^{-x^2}\}$$

$$= -\frac{1\,i\alpha}{2\sqrt{2}}\,e^{-\alpha^2/4}.$$

Problems in PDE's

Example 1 (a) $\frac{1}{k^2}u_t = u_{xx}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = f(x)$, $-\infty < x < \infty$

$\lim_{|x|\to\infty} u(x,t) = 0 = \lim_{|x|\to\infty} u_x(x,t) = 0$.

(b) Put $f(x) = U_0$ (constant) if $|x| < 1$

$= 0$ $\quad |x| > 0$.

Solution: Let $\mathcal{F}\{u\} = \hat{u}(\alpha, t)$

$\mathcal{F}\{u_{xx}\} = -\alpha^2\hat{u}(\alpha,t)$

$\mathcal{F}\{u_t\} = \frac{d\hat{u}}{dt}$

PDE $\Rightarrow$ $\frac{d\hat{u}}{dt} = -k^2\alpha^2\hat{u}$

This has solution $\hat{u}(\alpha,t) = C(\alpha)\,e^{-k^2\alpha^2 t}$

Using $\mathcal{F}\{u(x,0) = f(x)\} \Rightarrow \hat{u}(\alpha,0) = \hat{f}(\alpha)$, we get $C(\alpha) = \hat{f}(\alpha)$

So that $\hat{u}(\alpha,t) = \hat{f}(\alpha)\,e^{-k^2\alpha^2 t}$

Hence

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-k^2\alpha^2 t}\, e^{i\alpha x}\, d\alpha$$

using $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du$,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} e^{-k^2\alpha^2 t}\, e^{i(\alpha x - \alpha u)}\, d\alpha \right] du$$

We may use Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$ and property of even/odd functions to write

$$u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{0}^{\infty} e^{-k^2\alpha^2 t} \cos(\alpha x - \alpha u)\, d\alpha \right] du$$

(b) part can be solved by putting $\hat{f}(\alpha)$ in This Case.

Example (2) Wave Eqn

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty,\ t > 0$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = 0$$

$$u \to 0,\ u_x \to 0 \text{ as } |x| \to \infty, \quad t > 0$$

We take Fourier transform in x

$$\mathcal{F}\{u_{xx}\} = -\alpha^2 \hat{u}(\alpha,t)$$

$$\mathcal{F}\{u_{tt}\} = \hat{u}\ \frac{\partial^2 u}{\partial t^2}$$

$$\mathcal{F}\{u(x,0) = f(x)\} \Rightarrow \hat{u}(\alpha,0) = \hat{f}(\alpha)$$

$$\mathcal{F}\{u_t(x,0) = 0\} \Rightarrow \frac{\partial \hat{u}}{\partial t}(\alpha,0) = 0$$

The given problem transforms into

$$\frac{\partial^2 \hat{u}}{\partial t^2} + \alpha^2 c^2 \hat{u} = 0$$

The general solution is given by

$$\hat{u}(\alpha, t) = C_1 \cos\alpha c t + C_2 \sin \alpha c t$$

$$\frac{\partial \hat{u}}{\partial t}(\alpha, 0) = 0 \Rightarrow C_2 = 0.$$

$$\hat{u}(\alpha, 0) = \hat{f}(\alpha) \Rightarrow C_1 = \hat{f}(\alpha).$$

Hence $\hat{u}(\alpha, t) = \hat{f}(\alpha) \cos\alpha c t$

In order to invert, we use the fact that

$$e^{i\alpha ct} = \cos \alpha c t + i \sin \alpha c t \quad \text{and}$$

replace $\cos\alpha ct$ by $e^{\pm i\alpha ct}$ and then use

$\mathrm{Re}(e^{i\alpha ct})$ $\cos\alpha c t$ $= \frac{1}{2}(e^{i\alpha ct} + e^{-i\alpha ct})$

$$\hat{u}(\alpha, t) = \hat{f}(\alpha)\, e^{\pm i\alpha ct}, \quad \text{By shifting theorem}$$

$$\mathcal{F}^{-1}\{\hat{f}(\alpha)\, e^{\pm i\alpha ct}\} = f(x \pm ct)$$

Hence $\mathcal{F}^{-1}\{\hat{f}(\alpha) \cos\alpha ct\} = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$

Fourier Cosine and Fourier Sine Transforms

If $f(x)$ is defined for $x > 0$,

$$\left.\begin{aligned} \hat{f}_c(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\alpha x\, dx, \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\alpha) \cos\alpha x\, d\alpha \end{aligned}\right\} \quad \text{Fourier cosine transform pair}$$

$$\hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x \, dx,$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\alpha) \sin \alpha x \, d\alpha$$

Fourier Sine transform pair

Transforms of Derivatives

(a) $\mathcal{F}_c\{f'(x)\} = \alpha \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$

(b) $\mathcal{F}_c\{f''(x)\} = -\alpha^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

(c) $\mathcal{F}_s\{f'(x)\} = -\alpha \mathcal{F}_c\{f(x)\}$

(d) $\mathcal{F}_s\{f''(x)\} = -\alpha^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} \alpha f(0)$.

Remark: We can use Fourier cosine or sine derivatives if we have even order derivatives of the variable on which we take transform. Fourier cosine transform can be chosen if $f'(0)$ is given and Fourier sine transform if $f(0)$ is given.

Example. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$

$u(0,t) = 0$, $u(x,0) = f(x)$, $x > 0$

As suggested above, take Fourier sine transform in x.

$$\Rightarrow \frac{d}{dt} U_s(\alpha,t) = -\alpha^2 U_s(\alpha,t)$$

$$\Rightarrow U_s(\alpha,t) = A e^{-\alpha^2 t} \Rightarrow U_s(\alpha,t) = F_s(\alpha) e^{-\alpha^2 t}$$

$$\Rightarrow u(x,t) = \frac{2}{\pi} \int_0^\infty F_s(\alpha) e^{-\omega^2 t} \sin(\alpha x) \, d\alpha.$$