Chapter 7

Bessel's Equation and Bessel Functions

The differential equation

$$x^2 y'' + x y' + (\lambda x^2 - \nu^2) y = 0$$

is the Bessel equation of order ν, $\nu \geq 0$. In the Sturm - Liouville form is

$$(x y')' + \left(\lambda x - \frac{\nu^2}{x}\right) y = 0$$

This is a singular problem in interval $(0, l)$ where $r(x) = x = 0$ at one end point $x = 0$.

Thus we do not have a B.C. at $x = 0$.

According to Sturm - Liouville theorem, the solutions $J_\nu(x)$ are orthogonal with weight function x i.e.

$$\int_0^l x J_\nu(x) J_\mu(x)\, dx = 0, \quad \nu \neq \mu.$$

We shall first study the Bessel ~~funct~~ equations without eigenvalue λ.

Bessel Equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \qquad \text{— (1)}$$

As $x = 0$ is a regular singular point, assume

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r-2}$$

Putting in ①, we get

$$x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r-2} + x \sum_{n=0}^{\infty} c_n (n+r)\, x^{n+r-1} + x^2 \sum_{n=0}^{\infty} c_n x^{n+r} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

or

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r)\, x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

changing index so that x^{n+r} is the power of x in all terms, we get

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r)\, x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}\, x^{n+r} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Taking out $n=0,1$ terms and collecting others, we get

$$c_0\left[r(r-1) + r - \nu^2\right] x^r + c_1\left[(r+1)r + (r+1) - \nu^2\right] x + \sum_{n=2}^{\infty}\left[c_n\left\{(n+r)(n+r-1) + (n+r) - \nu^2\right\} + c_{n-2}\right] x^{n+r} = 0$$

$$\text{coeff}\,(x^r) = 0 \Rightarrow r^2 - \cancel{r} + \cancel{r} - \nu^2 = 0$$

$$\Rightarrow r^2 - \nu^2 = 0 \quad \text{or} \quad r = \pm\nu$$

$$\text{coeff}\,(x) = 0 \Rightarrow c_1 = 0$$ (we use $r = \pm\nu$ so $2\nu + 1 \neq 0$)

Coeff $x^{n+r} = 0 \Rightarrow$

$$C_n = -\frac{C_{n-2}}{(n+r)(n+r-1)+(n+r)-r^2}$$

For $\underline{r=\nu}$: the recurrence relation yields

$$C_n = -\frac{C_{n-2}}{n(n+2\nu)}$$

As $C_1 = 0$, so we obtain $C_1 = C_3 = C_5 = \cdots = 0$

when n is even ($n = 2n$)

$$C_{2n} = \frac{-1}{2n(2n+2\nu)} C_{2n-2} = -\frac{1}{2^2 n(n+\nu)} C_{2n-2}$$

This can be used to give

$$C_2 = -\frac{1}{2^2(\nu+1)} C_0$$

$$C_4 = \frac{-1}{2^2\, 2(\nu+2)} C_2$$

$$C_{2n-2} = \frac{-1}{2^2 (n-1)(n+\nu-1)} C_{2n-4}$$

$$C_{2n} = \frac{-1}{2^2 n(n+\nu)} C_{2n-2}$$

Taking product and cancelling repeated terms

$$C_{2n} = \frac{(-1)^n}{2^{2n} n!\,(1+\nu)\cdots(\nu+n)} C_0$$

The first solution is therefore

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{2n} (\nu+1)\cdots(\nu+n)} x^{2n+\nu}$$

The second solution will depend upon whether ν is an integer, zero or a rational fraction with $\nu_1 - \nu_2 \neq$ integer.

Gamma Function:

Gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt \quad , x > 0$$

If we put $x = n+1$, n a +ve integer

$$\Gamma(n+1) = \int_0^{\infty} t^n e^{-t} dt = \left[-t^n e^{-t}\right]_0^{\infty} - n\int_0^{\infty}(-e^{-t})\, t^{n-1} dt$$

$$= n\int_0^{\infty} t^{n-1} e^{-t} dt = n\,\Gamma(n).$$

Hence we can write

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1)$$
$$= n(n-1)\cdots 3\cdot 2\cdot 1 = n!$$

Thus $\boxed{\Gamma(n+1) = n!}$

In fact even if x is not an integer, we have

$$\left.\begin{aligned} &\Gamma(x+1) = x\,\Gamma(x) \quad , x > 0 \\ \text{or} \quad &\Gamma(x) = \frac{1}{x}\Gamma(x+1) \quad , x > 0 \end{aligned}\right\} \text{Factorial Property}$$

Example:

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)$$

Gamma Function for $-1<x<0$ and so on.

The above result $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$ can be used to define Γ function in $-1<x<0$.

For example $\Gamma\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}}\Gamma\left(-\frac{1}{2}+1\right) = -2\Gamma\left(\frac{1}{2}\right)$

This can be further carried out by using This property $\Gamma\left(-\frac{3}{2}\right) = \frac{1}{-\frac{3}{2}}\Gamma\left(-\frac{3}{2}+1\right) = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right)$

$$= \frac{4}{3}\Gamma(1/2).$$

Example: Consider $(1+\nu)(2+\nu)\cdots(n+\nu)$

using above property we find

$$\Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu) = (n+\nu)(n+\nu-1)\Gamma(n+\nu-1)$$

$$= \cdots\cdots (n+\nu)(n+\nu-1)\cdots\cdots(1+\nu)\Gamma(1+\nu)$$

Hence $(1+\nu)(2+\nu)\cdots(n+\nu) = \dfrac{\Gamma(n+\nu+1)}{\Gamma(1+\nu)}$

Hence we may write

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n\,\Gamma(\nu+1)}{2^{2n}\,n!\,\Gamma(n+\nu+1)} x^{2n+\nu}$$

If we choose $c_0 = \dfrac{1}{2^\nu\Gamma(1+\nu)}$, this solution is called the Bessel function of order ν of first kind

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu}\,n!\,\Gamma(n+\nu+1)} x^{2n+\nu}$$