Chapter 4

Convergence of Fourier series

Consider the Fourier series in $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

By pointwise convergence of such a series we mean that the partial sum

$$S_N(x) = \frac{1}{2} a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

satisfies

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \text{for a given value of } x.$$

Definition. A function f is said to be piecewise continuous on $[a,b]$ if

(a) f is defined and is continuous on all but a finite number of points on $[a,b]$

(b) the left and right ~~hand~~ limits exists at ~~end points~~ each point on $[a,b]$.

Re-call The left and right hand limits at x_0 are defined as

$$\lim_{\substack{x\to x_0 \\ x<x_0}} f(x) = f(x^-) \quad \text{and}$$

$$\lim_{\substack{x\to x_0^+ \\ x>x_0}} f(x) = f(x^+).$$

Example

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$f(x)$ is not continuous on $[-\pi, \pi]$ and is not defined at $x = 0$. $f(x)$ is piecewise continuous.

$$\lim_{x \to -\pi^+} f(x) = 1 \quad , \quad \lim_{x \to \pi^-} f(x) = 1$$

$$\lim_{\substack{x \to 0^+ \\ x > 0}} f(x) = 1, \quad \lim_{\substack{x \to 0^- \\ x < 0}} f(x) = -1.$$

Example $f(x) = \frac{1}{x}$ in $[-\pi, \pi]$

$$\lim_{\substack{x \to 0^+ \\ x > 0}} f(x) \quad \text{or} \quad \lim_{\substack{x \to 0^- \\ x < 0}} f(x)$$

doesnot exist.

This function is not piecewise continuous ~~at~~ in $[-\pi, \pi]$.

Remarks ① Every continuous function is also piecewise continuous.

② If $f(x)$ is piecewise continuous in $[-\pi, \pi]$ and it is periodic i.e. $f(x+2\pi) = f(x)$; then $f(x)$ is piecewise continuous for all x.

Pointwise Convergence of Fourier Series

If f and f' are piecewise continuous on $[-\pi, \pi]$, then the Fourier series in $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, \ldots$$

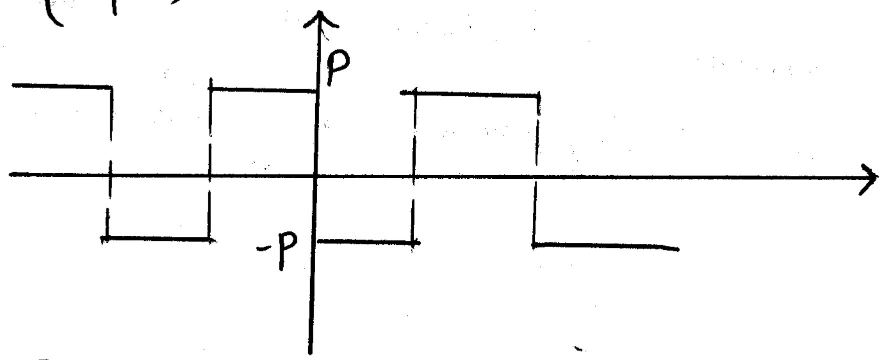
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \ldots$$

converges pointwise for all values of x in $[-\pi, \pi]$. The sum of the series equals $f(x)$ whenever $f(x)$ is continuous. At point of discontinuity it equals $\frac{f(x^+) + f(x^-)}{2}$.

(i.e. it gives average of L.H and R.H limits at such points).

Example: Consider

$$f(x) = \begin{cases} +p, & -\pi < x < 0 \\ -p, & 0 < x < \pi \end{cases} ; \quad f(x+2\pi) = f(x)$$



Square wave

$f(x)$ is an odd function, so a_n in the Fourier series will be zero.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} -p \sin nx \, dx$$

$$= \begin{cases} -\frac{4p}{n\pi}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

Therefore

$$f(x) = \frac{-4p}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2n-1)x}{2n-1}$$

(As only n odd terms are non zero).

Now we can see

$f(0) = 0$ from the series.

Also $\dfrac{f(0^-) + f(0^+)}{2} = \dfrac{p-p}{2} = 0.$

Interesting result:

$$f\left(\frac{\pi}{2}\right) = -p = -\frac{4p}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

or $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$

This gives relation of irrational number π with reciprocals of odd numbers.

Uniform Convergence

If given $\varepsilon > 0$, there exists N_0 (independent of x) on interval $[-\pi, \pi]$, such that

$$|f(x) - S_N(x)| < \varepsilon \quad \text{for all } N \geq N_0,$$

then we say $S_N(x)$ converges uniformly to $f(x)$ on $[-\pi, \pi]$ as $N \to \infty$.

The Fourier series of a continuous function f (of period 2π) converges uniformly to $f(x)$.

Example: Consider $f(x)$ in above example of square wave.

$$f(x) = +\frac{4P}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

$$f(x) = \begin{cases} -P, & -\pi < x \leq 0 \\ P, & 0 < x \leq \pi \end{cases}$$

If we allow ourself term by term differentiation then

$$0 = +\frac{4P}{\pi} \sum_{n=1}^{\infty} \cos(2n-1)x$$

This is absurd as for $x = 0$, it gives

$$0 = +\frac{4P}{\pi}(1+1+1+\cdots).$$

Thus term by term differentiation needs careful thought.

<u>Theorem</u>: If $f(x)$ is <u>continuous</u> with period 2π,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

with a_0, a_n, b_n given by formulae mentioned before, then term wise differentiation of series is possible

$$f'(x) = \sum_{n=1}^{\infty} n(-a_n \sin nx + b_n \cos nx)$$

which converges pointwise to $f'(x)$ whenever it is continuous.

<u>Theorem</u>: (Integration)

If f is a piecewise continuous function of period 2π, then the above Fourier series can be integrated term by term

$$\int_{-\pi}^{x} f(t)\,dt = \frac{1}{2} a_0 (x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n}[a_n \sin nx - b_n \cos nx + (-1)^n b_n]$$

<u>Example</u>

Consider $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$ $(*)$

where $f(x) = \begin{cases} -1, & -\pi < x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$

$f(x+2\pi) = f(x)$

Notice that if we take

$$g(x) = |x| = \begin{cases} -x, & -\pi < x \leq 0 \\ x, & 0 < x \leq \pi \end{cases}$$

Then $g'(x) = f(x)$ except at $x=0$.

~~Perf~~ Perform indefinite integration on $(*)$

$$g(x) = C - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

where C is constant of integration.

We Recognize it as $\frac{a_0}{2}$ in Fourier series,

so $C = \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|\, dx = \frac{\pi}{2}$.

Hence

$$g(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

<u>Exercise</u> Verify this by writing the Fourier series for $|x|$ in $[-\pi, \pi]$.