DEPARTMENT OF MATHEMATICAL SCIENCES

MATH 301 Methods of Applied Mathematics   Term o41

QUIZ # 1 (version 1)

Name ____________________ ID # __________ Sec # _____

Q1) Show that the following integral is independent of path. Evaluate it.

$$\int_{(-1,0)}^{(1,0)} (\sin x + e^{-y})dx + (y^2 - xe^{-y})dy .$$

I $\quad P(x,y) = \sin x + e^{-y}$

$\quad Q(x,y) = y^2 - x e^{-y}$

$\frac{\partial P}{\partial y} = -e^{-y}$, $\frac{\partial Q}{\partial x} = -e^{-y}$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Hence the integral is independent of path

II To find $\phi(x,y)$

$\frac{\partial \phi}{\partial x} = P = \sin x + e^{-y}$ — (1)

$\frac{\partial \phi}{\partial y} = Q = y^2 - x e^{-y}$ — (2)

Integrating (1) w.r.t. x

$\phi(x,y) = -\cos x + x e^{-y} + g(y)$

Diff. w.r.t. y

$\frac{\partial \phi}{\partial y} = -x e^{-y} + g'(y)$

Compare with (2)

$g'(y) = y^2$

or $g(y) = \frac{1}{3} y^3$

(ignoring constant c)

$\phi(x,y) = -\cos x + x e^{-y} + \frac{1}{3} y^3$

III The integral is

$\phi(1,0) - \phi(-1,0)$

$= -\cos(1) + 1 + 0 - (-\cos(-1) - 1 + 0)$

$= -\cancel{\cos 1} + 1 + \cancel{\cos 1} + 1 = 2$

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QUIZ # 1 (version 2)

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Q1) Show that $\underline{F}$ given below is a conservative field. Find the potential function $\phi(x,y,z)$

$$\underline{F}(x,y,z) = (2+e^{-z})\underline{i} + (2y-1)\underline{j} + (2-xe^{-z})\underline{k}.$$

I Here $P(x,y,z) = 2+e^{-z}$; $Q(x,y,z) = (2y-1)$; $R(x,y,z) = (2-xe^{-z})$

$\frac{\partial P}{\partial y} = 0$, $\frac{\partial P}{\partial z} = -e^{-z}$

$\frac{\partial Q}{\partial x} = 0$, $\frac{\partial Q}{\partial z} = 0$

$\frac{\partial R}{\partial x} = -e^{-z}$, $\frac{\partial R}{\partial y} = 0$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0$

$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 0$

$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = -e^{-z}$

Hence $\underline{F}$ is conservative and there exists a potential function $\phi(x,y,z)$

II $\frac{\partial \phi}{\partial x} = P = 2+e^{-z}$ — (1)

$\frac{\partial \phi}{\partial y} = Q = 2y-1$ — (2)

$\frac{\partial \phi}{\partial z} = R = 2-xe^{-z}$ — (3)

Integrate (1) w.r.t. x

$\phi(x,y,z) = 2x + xe^{-z} + f(y,z)$

Diff. w.r.t. y

$\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} = 2y-1$, using (2)

Integration gives $f(y,z) = y^2 - y + g(z)$

Thus $\phi(x,y,z) = 2x + xe^{-z} + y^2 - y + g(z)$

Diff w.r.t. z

$\frac{\partial \phi}{\partial z} = -xe^{-z} + g'(z)$

$= 2 - xe^{-z}$ using (3)

Thus $g'(z) = 2$

or $g(z) = 2z + C$

(we choose $C=0$).

$$\boxed{\phi(x,y,z) = 2x + xe^{-z} + y^2 - y + 2z}$$

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QUIZ # 1 (version 3)
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Q1) Check if the integral is independent of path. Evaluate

$$\int_{(0,1)}^{(3,4)} (x\sqrt{x^2+y^2})dx + (y\sqrt{x^2+y^2})dy$$

I $P = x\sqrt{x^2+y^2}$, $\frac{\partial P}{\partial y} = x\frac{1}{\cancel{2}}\cdot\cancel{2}y\,(x^2+y^2)^{-1/2}$

$Q = y\sqrt{x^2+y^2}$, $\frac{\partial Q}{\partial x} = y\cdot\frac{1}{\cancel{2}}\cancel{2}x\,(x^2+y^2)^{-1/2}$

Hence $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so the integral is independent of path

II There exists $\phi(x,y)$ such that

$\frac{\partial \phi}{\partial x} = P = x\sqrt{x^2+y^2}$ — (1)

$\frac{\partial \phi}{\partial y} = Q = y\sqrt{x^2+y^2}$ — (2)

Integrating (1) w.r.t x

$\phi(x,y) = \frac{1}{2}\int 2x\sqrt{x^2+y^2}\,dx$

$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}(x^2+y^2)^{3/2} + g(y)$

Differentiate w.r.t y

$\frac{\partial \phi}{\partial y} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(x^2+y^2)^{1/2}\cdot\cancel{2}y + g'(y)$

$= y\,(x^2+y^2)^{1/2} + g'(y)$

$= y\sqrt{x^2+y^2}$ using (2)

Hence we get

$g'(y) = 0$

or $g(y) = C$

We may chose $C=0$ without any loss.

$\phi(x,y) = \frac{1}{3}(x^2+y^2)^{3/2}$

Integral $= \phi(3,4) - \phi(0,1)$

$= \frac{1}{3}(9+16)^{3/2} - \frac{1}{3}(1)^{3/2}$

$= \frac{125}{3} - \frac{1}{3} = \frac{124}{3}$.