KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences

Math 301 Method of Applied Mathematics

Major Exam # 1 Term 041

Time Allowed 75 minutes

Name ____________________ ID # ________ Section # ____

Q #	Grade
1	/ 5
2	/ 4
3	/ 6
4	/ 5
Total	/ 20

Important Note

Show all work.
Use of programmable calculator is not allowed.
Mobiles and paging devices should not be carried during examination.

Instructor: F. D. Zaman

Q # 1(a) Find all points on $f(x,y) = 2x^3 - 24x + 3y^2 - 12y = 0$ for which $\|\underline{\nabla} f\| = 0.$ (2)

$$\underline{\nabla} f = \langle f_x, f_y \rangle = \langle 6x^2 - 24, 6y - 12 \rangle$$

$$\|\underline{\nabla} f\|^2 = (6x^2 - 24)^2 + (6y - 12)^2$$

$$\|\underline{\nabla} f\|^2 = 0 \Longrightarrow 6x^2 - 24 = 0 \quad , \quad 6y - 12 = 0$$

so that $6x^2 - 24 = 0 \Longrightarrow x = \pm 2$

$6y - 12 = 0 \Longrightarrow y = 2.$

Points are $(2,2)$, $(-2,2)$.

Q # 1(b) Evaluate the integral

$\int_C 2xydx - 4ydy + e^{xy}dz$ where C is the curve $x^2+4y^2=4$, $z = 2$. (3)

$C: \ x^2+4y^2=4, \ z=2$. Parametric form is

$$\left.\begin{array}{l} x = 2\cos t \\ 2y = 2\sin t \\ z = 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = 2\cos t \\ y = \sin t \\ z = 2 \end{array}\right\} \begin{array}{l} \frac{dx}{dt} = -2\sin t \\ \frac{dy}{dt} = \cos t \\ \frac{dz}{dt} = 0 \end{array}$$

$t = 0$ to 2π.

The given integral is

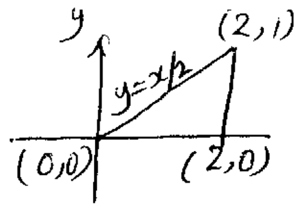
$$\int_0^{2\pi} [2(2\cos t)\sin t(-2\sin t) - 4(\sin t)\cos t + 0]\,dt$$

$$= \int_0^{2\pi} [-8\sin^2 t\cos t - 4\sin t\cos t]\,dt$$

$$= -\frac{8}{3}\left[\sin^3 t\right]_0^{2\pi} - \frac{4}{2}\left[\sin^2 t\right]_0^{2\pi} = 0.$$

Q # 2) Use Green's theorem to evaluate $\oint_C \underline{F}.d\underline{r}$ where $\underline{F} = \cos x^2 \underline{i} + x e^{y^2} \underline{j}$,

C being the triangle formed by vertices (0,0), (2,0) and(2,1). (4)



Here $P = \cos x^2$, $Q = x e^{y^2}$

$$\frac{\partial Q}{\partial x} = e^{y^2}, \quad \frac{\partial P}{\partial y} = 0.$$

Using Green's theorem the given integral is

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y} \right) dA = \iint_R e^{y^2} \, dx \, dy$$

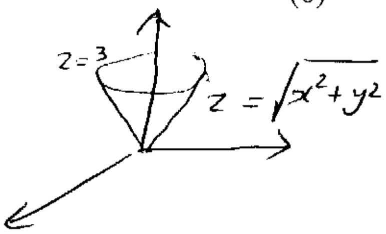
R: $x = 2y$ to $x = 2$; $y = 0$ to $y = 1$.

This gives $\displaystyle\int_0^1 \int_{2y}^{2} e^{y^2} \, dx \, dy = \int_0^1 \left[x e^{y^2} \right]_{2y}^{2} dy$

$$= \int_0^1 2 e^{y^2} \, dy - 1.$$

Q # 3) Consider a vector field $\underline{F}(x,y,z) = z\underline{i} + x\underline{j} + y\underline{k}$ and the surface formed by $z = \sqrt{x^2+y^2}$, $0 \le z \le 3$. Evaluate the following integrals and state the result that is verified by the evaluations.

$\oint_C \underline{F}.d\underline{r}$ **and** $\iint_S (curl\,\underline{F}).\underline{n}ds$ (6)



$C: \quad x^2+y^2 = 9 \quad (\text{at } z=3)$

$$\left.\begin{aligned} x &= 3\cos t \\ y &= 3\sin t \\ z &= 3 \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

$$\underline{F}(t) = 3\,\underline{i} + 3\cos t\,\underline{j} + 3\sin t\,\underline{k}$$

$$d\underline{r} = -3\sin t\,\underline{i} + 3\cos t\,\underline{j} + 0\,\underline{k}$$

$$\underline{F}.d\underline{r} = -9\sin t + 9\cos^2 t$$

$$\int_0^{2\pi} \underline{F}\cdot d\underline{r} = \int_0^{2\pi}\left[-9\sin t + \frac{9}{2}(1+\cos 2t)\right]dt$$

$$= \Big[+9\cos t\Big]_0^{2\pi} + \frac{9}{2}\Big[t\Big]_0^{2\pi} + \frac{9}{4}\Big[\sin 2t\Big]_0^{2\pi}$$

$$= 9\pi.$$

$$\iint_S (\text{Curl }\underline{F}).\underline{n}\,ds$$

$$\text{Curl }\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \underline{i} + \underline{j} + \underline{k}$$

$$f = z - \sqrt{x^2+y^2}$$

$$n = \frac{\nabla f}{\|\nabla f\|} = \frac{\left\langle -\frac{x}{\sqrt{x^2+y^2}}\underline{i},\ \frac{-y}{\sqrt{x^2+y^2}}\underline{j},\ \underline{k}\right\rangle}{\left(\frac{x^2+y^2}{x^2+y^2}+1\right)^{1/2}}$$

$$\text{Curl }\underline{F}\cdot\underline{n} = \frac{1}{\sqrt{2}}\left(1 - \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\sqrt{x^2+y^2} - x - y}{\sqrt{2}\ \sqrt{x^2+y^2}}$$

$$ds = \sqrt{1 + f_x^2 + f_y^2}\ dA \quad , \quad z = \sqrt{x^2+y^2}$$

$$ds = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\ dA$$

$$= \sqrt{2}\ dA.$$

We get $\int_0^{2\pi}\int_0^3 (1 - \cos\theta - \sin\theta)\, r\, dr\, d\theta$

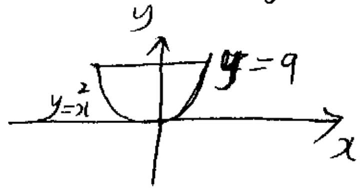
(in polar coordinates)

$$= 9\pi.$$

Q # 4) Use the Divergence theorem to evaluate the flux integral $\iint_S (\underline{F}.\underline{n})ds$ on the surface formed by $y = x^2$, $z = 9 - y$, $z = 0$, for the vector field given by $\underline{F} = 2x\underline{i} - e^z\,\underline{j} + z\,\underline{k}$. (5)

By Divergence theorem, the given integral is

x - y plane region



$$\iiint_V \operatorname{div} \underline{F}\, dV$$

$$\operatorname{div} \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2+1 = 3.$$

The volume is described by

$z = 0$ to $z = 9-y$

$y = x^2$ to $y = 9$ (see figure)

$x = -3$ to $x = 3$.

$$I = \int_{-3}^{3}\int_{x^2}^{9}\int_{0}^{9-y} 3\, dz\, dy\, dx$$

$$= 3\int_{-3}^{3}\int_{x^2}^{9} [z]_0^{9-y}\, dy\, dx = 3\int_{-3}^{3}\int_{x^2}^{9} (9-y)\, dy\, dx$$

$$= 3\int_{-3}^{3} \left[9y - \frac{y^2}{2}\right]_{x^2}^{9} dx = 3\int_{-3}^{3} \left[9x^2 - \frac{x^4}{2} - 81 + \frac{81}{2}\right] dx$$

$$= 3\left[3x^3 - \frac{x^5}{10} - \frac{81}{2}x\right]_{-3}^{3}$$

$$= 3\left[6\times 27 - \frac{3^5}{5} - 81(3)\right]$$