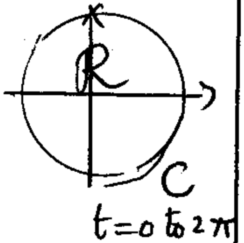
(9.12)

3) $\oint_C -y^2 dx + x^2 dy$

$= \iint_R (2x+2y)\, dA$

C: $x = 3\cos t,\ y = 3\sin t$



L.H.S.

$\frac{dx}{dt} = -3\sin t$

$\frac{dy}{dt} = 3\cos t$

We get

$\int_0^{2\pi} -9(\sin^2 t)(-3\sin t) + 9(\cos^2 t)\, 3\cos t\, dt$

$= 27 \int_0^{2\pi} \{(1-\cos^2 t)(\sin t) + (1-\sin^2 t)\cos t\}\, dt$

$= 27\left(-\cos t + \frac{1}{3}\cos^3 t + \sin t - \frac{1}{3}\sin^3 t\right)\Big|_0^{2\pi}$

$= 0.$

R.H.S. Use polar coordinates for region inside circle

$x = r\cos\theta,\ y = r\sin\theta \quad r = 0$ to 3, $\theta = 0$ to 2π

We get

$2\int_0^{2\pi}\int_0^3 (\cos\theta + \sin\theta)\, r\, dr\, d\theta$

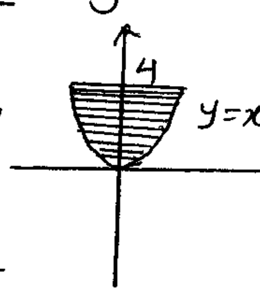
$= \cancel{2}\cdot\left[\frac{r^2}{\cancel{2}}\right]_0^3 \{\sin\theta - \cos\theta\}_0^{2\pi}$

$= 9 \cdot 0 \quad = 0.$

25) $P = -\frac{y^3}{(x^2+y^2)^2},\quad Q = \frac{xy^2}{(x^2+y^2)^2}$

$\frac{\partial P}{\partial y} = \frac{-3y^2(x^2+y^2)^2 + 4y^4(x^2+y^2)}{(x^2+y^2)4} = \frac{-3y^2(x^2+y^2)+4y^4}{(x^2+y^2)4}$

6) $\oint_C (x^2+y^2)dx + (2x^2-y)dy$



$P = x^2+y^2,\ Q = 2x^2-y$

C: $y = x^2,\ y = 4$

By Green's theorem the integral is

$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\right) dA$

over region: $y = x^2$ to $y = 4$

$x = -2$ to $x = 2$

This gives

$\int_{-2}^{2}\int_{y=x^2}^{4} (4x - 2y)\, dy\, dx = -\frac{96}{5}$

(18) $\frac{1}{2}\oint_C -y\,dx + x\,dy =$ Area of region

Here $P = -\frac{1}{2}y,\ Q = \frac{1}{2}x$

$\frac{\partial P}{\partial y} = -\frac{1}{2},\quad \frac{\partial Q}{\partial x} = \frac{1}{2}$

By Green's theorem, the integral is $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dA$

$= \iint_R dA$

$=$ Area of R

$\frac{\partial Q}{\partial x} = \frac{y^2(x^2+y^2)^2 - 4x^2y^2(x^2+y^2)}{(x^2+y^2)^4}$

$= \frac{y^2(x^2+y^2) - 4x^2y^2}{(x^2+y^2)^4}$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Choose $x^2+y^2 = 1$ as curve of integration.

Put $x = \cos t,\ y = \sin t$.

3) Surface area of portion of cylinder $x^2+z^2=16$ that is above first quadrant bounded by $x=0,\ x=2,\ y=0,\ y=5$.

$z=f(x,y)=\sqrt{16-x^2}$, $z\geq 0$.

to find ds

$f_x = -x(16-x^2)^{-1/2}$

$f_y = 0$

~~dsco~~ $\sqrt{1+f_x^2+f_y^2} = \sqrt{1+x^2(16-x^2)^{-1}}$

$\sqrt{1+\frac{x^2}{16-x^2}} = \frac{4}{\sqrt{16-x^2}}$

$\therefore ds = \frac{4}{\sqrt{16-x^2}}\, dA.$

Surface Area $= \int_0^2\int_0^5 \frac{4}{\sqrt{16-x^2}}\, dy\, dx$

$= 4\times 5\int_0^2 \frac{dx}{\sqrt{16-x^2}}$

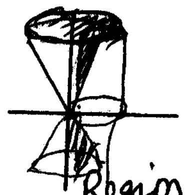
$= 20\left[\sin^{-1}\frac{x}{4}\right]_0^2 = 10\,\pi/3.$

10) Surface area of portions of cone

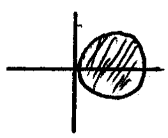
$z^2 = \frac{1}{4}(x^2+y^2)$

within cylinder ~~...~~

$(x-1)^2+y^2=1$



Region



Here $z = \frac{1}{2}\sqrt{x^2+y^2}$

$f_x = \frac{x}{2}(x^2+y^2)^{-1/2}$

$f_y = \frac{y}{2}(x^2+y^2)^{-1/2}$

$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+\frac{x^2}{4(x^2+y^2)}+\frac{y^2}{4(x^2+y^2)}}$

$= \sqrt{\frac{5(x^2+y^2)}{4(x^2+y^2)}} = \frac{\sqrt{5}}{2}$

Region: $x^2+y^2-2x=0 \Rightarrow r^2-2r\cos\theta=0$

$ds = \frac{5}{2}\, dA$

$\Rightarrow r=2\cos\theta$
We have
$r=0$ to $2\cos\theta$

Hence

$S = 2\left[2\int_0^{\pi/2}\int_0^{2\cos\theta}\frac{\sqrt{5}}{2}\, r\, dr\, d\theta\right.$

$= \frac{\cancel{2}4\sqrt{5}}{2}\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$

*We have an upper and a lower cone

$= 4\sqrt{5}\int_0^{\pi/2}\cos^2\theta\, d\theta$

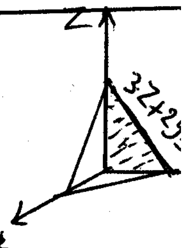
$= 2\sqrt{5}\int_0^{\pi/2}(1+\cos 2\theta)\, d\theta$

$= 2\sqrt{5}\left(\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= \cancel{2}\frac{\sqrt{5}}{\cancel{2}}\pi = \sqrt{5}\,\pi.$

26) The surface is

$x = g(y,z) = 6-2y-3z$



$g_y=-2,\ g_z=-3$

$\sqrt{1+g_y^2+g_z^2} = \sqrt{1+4+9} = \sqrt{14}$

Surface integral using yz-plane and $x=6-2y-3z$ gives

$\sqrt{14}\int_0^2\int_0^{3-3z/2}(3z^2+4yz)\, dy\, dz$

$= \sqrt{14}\int_0^2\left[3yz^2+2y^2z\right]_{y=0}^{3-\frac{3z}{2}} dz$

$= \sqrt{14}\int_0^2\left\{3z^2\left(3-\frac{3z}{2}\right)+2z\left(3-\frac{3}{2}z\right)^2\right\}dz$

simplify and integrate