6) $\mathcal{L}\{t e^{-3t}\cos 3t\}$

Use the properties

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{t\ f(t)\} = -\frac{d}{ds}(F(s))$

Now $\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$

So $\mathcal{L}\{e^{-3t}\cos 3t\} = \frac{s+3}{(s+3)^2+9}$

and $\mathcal{L}\{t e^{-3t}\cos 3t\} = -\frac{d}{ds}\left[\frac{s+3}{(s+3)^2+9}\right]$

$= \frac{(s+3)^2-9}{[(s+3)^2+9]^2}$ (Check!)

16) $\mathcal{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\}$

By convolution property

$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = F(s)G(s)$

Here $f(t)=\sin t \Rightarrow F(s)=\frac{1}{s^2+1}$

$g(t)=\cos t \Rightarrow G(s)=\frac{s}{s^2+1}$

Hence $\mathcal{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right. = \frac{s}{(s^2+1)^2}$.

19)(c) $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s-1)}\right\}$

Let us write $\frac{1}{s^3(s-1)} = F(s)G(s) = \frac{1}{s^3}\cdot\frac{1}{s-1}$

$F(s)=\frac{1}{s^3} \Rightarrow f(t)=\frac{1}{2}t^2$

$G(s)=\frac{1}{s-1} \Rightarrow g(t)=e^t$

Thus $\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$

$= \frac{1}{2}\int_0^t (\tau^2 e^{t-\tau})\,d\tau$

$= \frac{1}{2}e^t \underbrace{\int_0^t \tau^2 e^{-\tau}\,d\tau}_{\text{integrate by parts}}$

$= \frac{1}{2}e^t\left[[-\tau^2 e^{-\tau}]_0^t + 2\int_0^t \tau e^{-\tau}\,d\tau\right]$

$= \frac{1}{2}e^t\left[-t^2e^{-t} + 2\{[-\tau e^{-\tau}]_0^t - [e^{-\tau}]_0^t\}\right]$

$= \frac{1}{2}e^t\left[-t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2\right]$

$= -\frac{1}{2}t^2 - t - 1 + e^t$.

34) $y''+y=f(t)$, $y(0)=1$, $y'(0)=0$,

$f(t) = \begin{cases} 1, & 0\le t<\pi/2 \\ \sin t, & t\ge \pi/2 \end{cases}$

We can write

$f(t) = 1 - (1-\sin t)\,u(t-\frac{\pi}{2})$

$= 1 - u(t-\frac{\pi}{2}) + \sin t\,u(t-\frac{\pi}{2})$

$= 1 - u(t-\frac{\pi}{2}) + \cos(t-\frac{\pi}{2})\,u(t-\frac{\pi}{2})$

Using $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$F(s) = \frac{1}{s} - \frac{e^{-\frac{\pi s}{2}}}{s} + \frac{s}{s^2+1}e^{-\frac{\pi s}{2}}$.

Taking Laplace transform D.E ⇒ (use usual way)

$s^2Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s} - \frac{e^{-\frac{\pi s}{2}}}{s} + \frac{s}{s^2+1}e^{-\frac{\pi s}{2}}$

or $Y(s) = \frac{s}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s(s^2+1)}e^{-\frac{\pi s}{2}} + \frac{s}{(s^2+1)^2}e^{-\frac{\pi s}{2}}$

Inversion ⇒ (Check calculations)

$y(t) = 1 - (1-\sin t)\,u(t-\frac{\pi}{2}) + \frac{1}{2}(t-\frac{\pi}{2})\cos t\,u(t$

Main result: $\mathcal{L}\{\delta(t-a)\} = e^{-sa}$.

(8) $y'' - 2y' = 1 + \delta(t-2)$

$y(0) = 0,\ y'(0) = 1.$

$\mathcal{L}\{\delta(t-2)\} = e^{-2s}$

$\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - y(0)$

$= s\,Y(s)$

$\mathcal{L}\{y''\} = s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0)$

$= s^2\,Y(s) - 1.$

We thus get from given problem

$$s^2 Y(s) - 1 - 2s\,Y(s) = \frac{1}{s} + e^{-2s}$$

or $$s(s-2)\,Y(s) = 1 + \frac{1}{s} + e^{-2s}$$

or $$Y(s) = \frac{1}{s(s-2)} + \frac{1}{s^2(s-2)} + \frac{e^{-2s}}{s(s-2)}$$

Now $$\frac{1}{s(s-2)} \equiv \frac{1}{2}\left\{\frac{1}{s-2} - \frac{1}{s}\right\}$$ (Partial fractions)

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} = \frac{1}{2}e^{2t} - \frac{1}{2} \quad \text{—(1)}$$

Also using $\mathcal{L}^{-1}\{F(s)\,e^{-as}\} = f(t-a)\,\mathcal{U}(t-a)$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s-2)}\right\} = \left(\frac{1}{2}e^{2(t-1)} - \frac{1}{2}\right)\mathcal{U}(t-2).$$

$a = -2$ —(2)

Also, $$\frac{1}{s^2(s-2)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$$

$$\equiv -\frac{1}{4}\cdot\frac{1}{s} - \frac{1}{2}\,\frac{1}{s^2} + \frac{1}{4}\,\frac{1}{s-2}$$

$$\mathcal{L}^{-1}\frac{1}{s^2(s-2)} = \frac{1}{4}e^{2t} - \frac{1}{4} - \frac{1}{2}t \quad \text{—(3)}$$

From (1), (2), (3)

$$y(t) = \frac{3}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + \frac{1}{2}\left(e^{2(t-1)} - 1\right)\mathcal{U}(t-2).$$

12) $y'' - 7y' + 6y = e^t + \delta(t-2) + \delta(t-4)$,

$y(0) = 0,\ y'(0) = 0$

$\mathcal{L}\{e^t\} = \frac{1}{s-1}$

$\mathcal{L}\{\delta(t-2)\} = e^{-2s}$

$\mathcal{L}\{\delta(t-4)\} = e^{-4s}$

D.E $\Rightarrow$

$$s^2 Y(s) - 7s\,Y(s) + 6Y(s) = \frac{1}{s-1} + e^{-2s} + e^{-4s}$$

or $$\underbrace{(s^2 - 7s + 6)}_{(s-1)(s-6)}\,Y(s) = \frac{1}{s-1} + e^{-2s} + e^{-4s}$$

$$Y(s) = \frac{1}{(s-1)^2(s-6)} + \frac{e^{-2s}}{(s-1)(s-6)} + \frac{e^{-4s}}{(s-1)(s-6)}$$

Now $$\frac{1}{(s-1)^2(s-6)} \equiv -\frac{1}{25}\,\frac{1}{s-1} - \frac{1}{5}\,\frac{1}{(s-1)^2}$$

$$\frac{1}{(s-1)(s-6)} \equiv -\frac{1}{5}\cdot\frac{1}{s-1} + \frac{1}{5}\cdot\frac{1}{s-6}$$

$$\mathcal{L}^{-1}\frac{1}{(s-1)^2(s-6)} = -\frac{1}{25}e^t - \frac{1}{5}te^t \quad \text{—(1)}$$

As $$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-6)}\right\} = -\frac{1}{5}e^t + \frac{1}{5}e^{6t}$$

so $$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s-1)(s-6)}\right\} = \left(-\frac{1}{5}e^{t-2} + \frac{1}{5}e^{6(t-2)}\right)\mathcal{U}(t-2) \quad \text{—(2)}$$

and $$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{(s-1)(s-6)}\right\} = \left(-\frac{1}{5}e^{t-4} + \frac{1}{5}e^{6(t-4)}\right)\mathcal{U}(t-4) \quad \text{—(3)}$$

(1) + (2) + (3) gives $y(t)$.