⑥ $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \cos t, & t \ge \pi/2 \end{cases}$

$$I = \mathcal{L}\{f(t)\} = \int_{\pi/2}^{\infty} \cos t \; e^{-st}\, dt$$

$$= \underbrace{\left[\cos t \frac{e^{-st}}{-s}\right]_{\pi/2}^{\infty}}_{=0} - \frac{1}{s}\int_{\pi/2}^{\infty} \sin t \; e^{-st}\, dt$$

$$= -\frac{1}{s}\left[\left[\sin t \frac{e^{-st}}{-s}\right]_{\pi/2}^{\infty} + \frac{1}{s}\int_{\pi/2}^{\infty} \cos t \, e^{-st} dt \right]$$

$$I = -\frac{e^{-s\pi/2}}{s^2} - \frac{1}{s^2} I$$

or $\left(1 + \frac{1}{s^2}\right) I = -\frac{e^{-s\pi/2}}{s^2}$

$\left(\frac{s^2+1}{s^2}\right) I = -e^{-s\pi/2}/s^2$

or $I = -\frac{e^{-s\pi/2}}{s^2+1}$

㊳ $f(t) = \cos^2 t$

$= \frac{1}{2}(1 - \cos 2t)$

$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\mathcal{L}(1) - \mathcal{L}(\cos 2t)\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$

㉚ $f(t) = (e^t - e^{-t})^2$

$= e^{2t} + e^{-2t} - 2$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{2t}\} + \mathcal{L}\{e^{-2t}\} - 2\mathcal{L}\{1\}$

$= \frac{1}{s-2} + \frac{1}{s+2} - \frac{2}{s}.$

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40) a

We have $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0.$ — ①

For $f(t) = t^{-1/2}$

$\mathcal{L}\{f(t)\} = \int_0^{\infty} t^{-1/2} e^{-st} dt$ — ②

In order to use ① we put

$st = u \Rightarrow s\,dt = du$

$t = 0 \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

(2) gives

$$\int_0^{\infty} \frac{u^{-\frac{1}{2}}}{s^{-1/2}} e^{-u} \frac{du}{s}$$

$$= \frac{1}{\sqrt{s}} \int_0^{\infty} u^{-1/2} e^{-u}\, du.$$

From ① we find ($\alpha = 1/2$)

$\mathcal{L}\{f(t)\} = \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right).$

As $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\mathcal{L}\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}}.$

(5) $\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s^3+3s^2+3s+1}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}+\frac{1}{s^4}\right\}$

$= 1+3t+\frac{3}{2}t^2+\frac{1}{6}t^3.$

(16) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$

$= \cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}\sin\sqrt{2}\,t.$

(35) $y''+5y'+4y=0, \quad y(0)=1, \quad y'(0)=0$

$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0)$

$= s^2 Y(s) - s.$

$\mathcal{L}\{y'\} = s\,Y(s) - y(0) = s\,Y(s) - 1.$

Thus the Laplace transform of the problem gives

$s^2 Y(s) - s + 5s\,Y(s) - 5 + 4Y(s) = 0$

or $(s^2+5s+4)Y(s) = 5+s$

or $Y(s) = \frac{5+s}{s^2+5s+4} = \frac{5+s}{(s+1)(s+4)} \equiv \frac{4}{3}\cdot\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}$

(By partial fractions)

Thus $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{3}\cdot\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}\right\}$

$= \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}.$

8) $\mathcal{L}\{e^{-2t}\cos 4t\}$

As $\mathcal{L}\{\cos 4t\} = \dfrac{s}{s^2+16}$.

By shifting on s-axis

$$\mathcal{L}\{e^{-2t}\cos 4t\} = \frac{s+2}{(s+2)^2+16}$$

(13) $$\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$$

$$\frac{1}{s^2-6s+10} = \frac{1}{(s-3)^2+1}$$

Now $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\} = \sin t \; e^{3t}.$$

(20) $$\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}.$$

In order to use shifting on s-axis, we write

$$\frac{(s+1)^2}{(s+2)^4} = \frac{s^2+2s+1}{(s+2)^4}$$

$$= \frac{s(s+2)}{(s+2)^4} + \frac{1}{(s+2)^4}$$

$$= \frac{s}{(s+2)^3} + \frac{1}{(s+2)^4}$$

$$= \frac{s+2-2}{(s+2)^3} + \frac{1}{(s+2)^4}$$

$$= \frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{(s+2)^4}$$

Now we can use shifting Theore
to get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{(s+2)^4}\right\}$$

$$= t e^{-2t} - t^2 e^{-2t} + \frac{1}{6} e^{-2t}.$$

(24) $y'' - 4y' + 4y = t^3 e^{2t}$ $y(0)=y'(0)=$

As $\mathcal{L}\{t^3\} = \dfrac{6}{s^4}$

so $\mathcal{L}\{t^3 e^{2t}\} = \dfrac{6}{(s-2)^4}$.

$$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0)$$
$$= s^2 Y(s)$$

$\mathcal{L}\{y'\} = s\,Y(s) - y(0) = sY($

Thus L.T. of problem gives

$s^2 Y(s) - 4s\,Y(s) + 4Y(s) = \dfrac{6}{(s-}$

or $(s^2-4s+4)Y(s) = \dfrac{6}{(s-2)}$

or $$Y(s) = \frac{6}{(s-2)^4(s-2)^2}$$

or $$Y(s) = \frac{6}{(s-2)^6}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$= \frac{6}{5!} t^5 e^{2t}.$$

$$= \frac{1}{20} t^5 e^{2t}.$$

Re-call

$\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$.

38) $\mathcal{L}\{e^{2-t} u(t-2)\}$

$= \mathcal{L}\{e^{-(t-2)} u(t-2)\}$.

$f(t) = e^{-t}$, $F(s) = \dfrac{1}{s+1}$

Hence $\mathcal{L}\{e^{-(t-2)} u(t-2)\} = \dfrac{e^{-2s}}{s+1}$.

47) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$.

Firstly, $\dfrac{1}{s(s+1)} \equiv \dfrac{1}{s} - \dfrac{1}{s+1}$

So $F(s) = \dfrac{1}{s(s+1)} \Rightarrow f(t) = 1 - e^{-t}$

$\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$ here $a = 1$, $t \to t-1$

$= (1 - e^{-(t-1)})\, u(t-1)$.

66) $y'' + 4y = f(t)$, $y(0) = 0$, $y'(0) = -1$

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \geq 1. \end{cases}$$

Write $f(t) = 1 - u(t-1)$

$\mathcal{L}\{f(t)\} = \dfrac{1}{s} - \dfrac{e^{-s}}{s}$

$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0)$
$= s^2 Y(s) + 1$

The transformed problem is

$(s^2+4) Y(s) = -1 + \dfrac{1}{s} - \dfrac{e^{-s}}{s}$

or $Y(s) = \dfrac{-1}{s^2+4} + \dfrac{1}{s(s^2+4)} - \dfrac{e^{-s}}{s(s^2+4)}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\} = \dfrac{1}{2} \sin t$.

By partial fractions

$F(s) = \dfrac{1}{s(s^2+4)} \equiv \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4}$

$1 \equiv A(s^2+4) + (Bs+C)s$

$s = 0 \Rightarrow A = 1/4$

Coff $s^2 \Rightarrow A + B = 0 \Rightarrow B = -\dfrac{1}{4}$

Coeff $s \Rightarrow C = 0$

$\therefore F(s) = \dfrac{1}{s(s^2+4)} \equiv \dfrac{1}{4}\dfrac{1}{s} - \dfrac{1}{4}\dfrac{s}{s^2+4}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\} = \dfrac{1}{4} - \dfrac{1}{4}\cos 2t$.

Also by shifting on t-axis

$\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s^2+4)}\right\}_{t \to t-1} = \left\{\dfrac{1}{4} - \dfrac{1}{4}\cos 2(t-1)\right\} u(t-1)$

Hence

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\dfrac{1}{2}\sin t$

$+ \dfrac{1}{4} - \dfrac{1}{4}\cos 2t - \left\{\dfrac{1}{4} - \dfrac{1}{4}\cos 2(t-1)\right\}$

$* \, u(t-1)$