(11) We need to show that

$$\int_0^p 1.\cos\frac{n\pi}{p}x\,dx = 0, \quad n = 1,2,3\ldots$$

and $$\int_0^p \cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx = 0 \quad m \neq n$$

For the first, we integrate to get

$$\frac{p}{n\pi}\left[\sin\frac{n\pi}{p}x\right]_0^p = 0.$$

Using $\cos x \cos y = \frac{1}{2}\{\cos(x-y) + \cos(x+y)\}$, the second integral gives

$$\int_0^p \cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx = \frac{1}{2}\int_0^p\left\{\cos\frac{(m-n)\pi}{p}x + \cos\frac{(m+n)}{p}\pi x\right\}dx$$

$$= \left[\frac{p}{2(m-n)}\sin\frac{(m-n)\pi}{p}x\right]_0^p + \left[\frac{p}{2(m+n)}\sin\frac{(m+n)\pi}{p}x\right]_0^p = 0.$$

To find norm,

$$\left\|\cos\frac{n\pi}{p}x\right\|^2 = \int_0^p \cos^2\frac{n\pi}{p}x\,dx$$

$$= \int_0^p\left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx$$

$$= \left[\frac{1}{2}x\right]_0^p + \left[\frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\right]_0^p = \frac{p}{2}.$$

Also $\|1\|^2 = \int_0^p 1.\,dx = p.$

16) $\phi_0(x) = 1$, $\phi_1(x) = x$. Thus

$$\int_a^b \phi_n(x).1\,dx = 0, \quad n > 2$$

$$\int_a^b \phi_n(x).x\,dx = 0, \quad n > 2$$

$(n = 0,\ \phi_0 = 1,\ n = 1,\ \phi_1 = x)$

Thus $$\int_a^b (\alpha x + \beta)\phi_n(x) = \alpha\int_a^b x\phi_n(x)\,dx + \beta\int_a^b \phi_n(x)\,dx = 0 \quad n > 2.$$

(18) $f_1(x) = x$, $f_2(x) = x^2$.

$f_1(x) = x + C_1x^2 + C_2x^3$.

$f_1(x)$, orthogonal to $x \Rightarrow$

$$\int_{-2}^{2} x(x + C_1x^2 + C_2x^3)\,dx = 0$$

$$\Rightarrow \left[\frac{x^3}{3} + C_1\frac{x^4}{4} + C_2\frac{x^5}{5}\right]_{-2}^{2} = 0$$

$$\Rightarrow \left[\frac{1}{3} + \frac{C_1}{4}x + C_2\frac{x^2}{5}\right]_{-2}^{2} = 0 \quad \text{--- (1)}$$

$f_1(x)$ orthogonal to $x^2 \Rightarrow$

$$\int_{-2}^{2} x^2(x + C_1x^2 + C_2x^3)\,dx = 0$$

$$\Rightarrow \left[\frac{x^4}{4} + \frac{C_1}{5}x^5 + C_2\frac{x^6}{6}\right]_{-2}^{2} = 0$$

$$\Rightarrow \left[\frac{1}{4} + \frac{1}{5}C_1x + C_2\frac{x^2}{6}\right]_{-2}^{2} = 0 \quad \text{--- (2)}$$

From (1) and (2)

$$\frac{16}{3} + \frac{64}{5}C_2 = 0$$

and $\frac{64}{5}C_1 = 0 \Rightarrow C_1 = 0$

so $C_2 = -\frac{16}{3}\times\frac{5}{64} = -\frac{5}{12}.$

6) $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi. \end{cases}$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\,dx + \frac{1}{\pi}\int_0^{\pi} (\pi^2 - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\pi^2 x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{5}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2 \cos nx\,dx$$

$$+ \frac{1}{\pi}\int_0^{\pi} (\pi^2 - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left[\frac{\pi^2 - x^2}{n}\sin nx\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi} x \sin nx\,dx\right]$$

$$= \frac{2}{n^2}(-1)^{n+1}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2 \sin nx\,dx$$

$$+ \frac{1}{\pi}\int_0^{\pi} (\pi^2 - x^2)\sin nx\,dx$$

$$= \frac{\pi}{n}\left[\left[(-1)^n - 1\right] + \frac{1}{\pi}\left(\frac{x^2 - \pi^2}{n}\cos n\pi\right)_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}\left[1 - (-1)^n\right]$$

Thus

$$f(x) = \frac{5\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^{n+1}\cos nx + \left(\frac{\pi}{n}(-1)^n + \frac{2\left[1 - (-1)^n\right]}{n^3\pi}\right)\sin nx\Bigg].$$

16) In this case

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} (e^x - 1)\,dx$$

$$= \frac{1}{\pi}\left(e^{\pi} - \pi - 1\right)$$

Similarly

$$a_n = \frac{1}{\pi}\int_0^{\pi} (e^x - 1)\cos nx\,dx$$

Integration by parts $\Rightarrow$

$$a_n = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

$$b_n = \frac{(-1)^n\, n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}$$

(Details omitted, Please work out,

20) Problem (7) $\Rightarrow$

$$f(x) = x + \pi = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$

Put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin\frac{n\pi}{2}$$

$$= \pi + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

Hence

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

14) $f(x) = x, \quad -\pi < x < \pi.$

$f(x)$ is odd and so $a_0 = a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{-x}{n}\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}$$

$$= -\frac{2}{n}\cos n\pi + \frac{1}{n^2}\cdot\frac{2}{\pi}\left[\cancel{\sin nx}\right]_0^{\pi} \quad =0$$

$$= -\frac{2}{n}(-1)^{n+1}.$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx.$$

(16) $f(x) = x|x|$ is odd so we get a sine series, $-1 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \int_{-1}^{1} x|x| \sin n\pi x\,dx$$

$$= 2\int_0^1 x^2 \sin n\pi x\,dx$$

$$= 2\left[\left[-\frac{x^2}{n\pi}\cos n\pi x\right]_0^1 + \frac{2}{n\pi}\int_0^1 x\cos n\pi x\,dx\right]$$

$$= 2\,\frac{(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right]$$

26) $f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x < 1. \end{cases}$

Fourier Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx, \quad n = 0, 1, 2, 3\ldots$$

$a_0 = 1.$

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = 2\int_{\frac{1}{2}}^{1} 1\cdot\cos n\pi x\,dx$$

$$= 2\left[\frac{1}{n\pi}\sin n\pi x\right]_{1/2}^{1} = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

Fourier Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = 2\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\int_{1/2}^{1} 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} + (-\right.$$

38) Fourier series in $(0, 2)$

$$a_0 = \frac{2}{2}\int_0^2 (2-x)\,dx = 2$$

$$a_n = \frac{2}{2}\int_0^2 (2-x)\cos n\pi x\,dx$$

$$= 2\left[\left[\frac{(2-x)}{n\pi}\cancel{\sin n\pi x}\right]_0^2 + \frac{1}{n\pi}\int_0^2 \sin n\pi x\right.$$

$$= \frac{2}{n\pi}\left(\frac{-1}{n\pi}\right)\left[\cos n\pi x\right]_0^2 = 0$$

$$b_n = 2\int_0^2 (2-x)\sin n\pi x\,dx$$

$$= 2\left[\left[-\frac{(2-x)}{n\pi}\cos n\pi x\right]_0^2 - \frac{1}{n\pi}\int_0^2 \cos n\pi x\,dx\right] \quad = 0$$

$$= \frac{4}{n\pi}$$

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n\pi}\sin n\pi x.$$