(1.1)

11) Verify by substitution:

$$x^2 y'' + 5xy' + 4y = 0; \; y_1 = \frac{1}{x^2}$$

$$y_2 = \frac{\ln x}{x^2}$$

Consider $y_1 = \frac{1}{x^2}$, then $y_1' = -\frac{2}{x^3}$

$y_1'' = \frac{6}{x^4}$. Put these in the D.E

$$x^2\left(\frac{6}{x^4}\right) + 5x\left(-\frac{2}{x^3}\right) + 4\left(\frac{1}{x^2}\right)$$

$$= \frac{6}{x^2} - \frac{10}{x^2} + \frac{4}{x^2} = 0.$$

Hence $y_1 = \frac{1}{x^2}$ is a solution

Now, $y_2 = \frac{\ln x}{x^2}$ gives $y_2' = \frac{x^2 \cdot \frac{1}{x} - 2x \ln x}{x^4}$

$$y_2' = \frac{1}{x^3} - \frac{2}{x^3} \ln x$$

$$y_2'' = -\frac{3}{x^4} + \frac{6}{x^4} \ln x - \frac{2}{x^4}$$

Put in D.E $\Rightarrow$

$$x^2\left(-\frac{5}{x^4} + \frac{6}{x^4} \ln x\right) + 5x\left(\frac{1}{x^3} - \frac{2}{x^3} \ln x\right) + 4\frac{\ln x}{x^2}$$

$$= -\frac{5}{x^2} + \frac{6}{x^2} \ln x + \frac{5}{x^2} - \frac{2}{x^2} \ln x + \frac{4 \ln x}{x^2}$$

$$= 0.$$

Hence y_2 is also a solution.

20) Verify and determine C:

$y' = x - y$; $y(x) = Ce^{-x} + x - 1$,

$y(0) = 10$.

We have $y(x) = Ce^{-x} + x - 1$

so, $y'(x) = -Ce^{-x} + 1$

D.E $\Rightarrow$ $-Ce^{-x} + 1 = \cancel{x} - Ce^{-x} - \cancel{x} + 1$

Hence $y(x)$ is indeed a solution.

Now use $y(0) = 10$

This gives

$10 = C - 1 \Rightarrow C = 11.$

The solution: $y(x) = 11e^{-x} + x - 1.$

30) The graph of g is normal to curves $y = x^2 + k$.

If two curves are normal then their slopes satisfy $m_1 m_2 = -1$ (slopes mean slopes of tangent at that point).

Now slop of given curves

$$m_2 = \frac{dy}{dx} = 2x.$$

Therefore $m_1 = -\frac{1}{2x}$.

Thus $\frac{dy}{dx} = -\frac{1}{2x}$

is the DE of required curve.

39) $xy' + y = 3x^2$.

We expect to be a multiple of x^2 (as $y + xy'$ gives $3x^2$)

Assume (guess) $y = Ax^2$

$y' = 2Ax$ then

D.E. $\Rightarrow$

$$x(2Ax) + Ax^2 = 3x^2$$

or $3Ax^2 = 3x^2$

L.H.S is identical to RHS if

$A = 1$.

Thus $y = x^2$ is a solution

6) $\dfrac{dy}{dx} = x\sqrt{x^2+9}$, $y(-4) = 0$.

Integration gives

$$y = \int x\sqrt{x^2+9}\,dx = \frac{1}{2}\int 2x\sqrt{x^2+9}\,dx$$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}(x^2+9)^{\frac{3}{2}} + C$$

$$\Rightarrow y = \frac{1}{3}(x^2+9)^{\frac{3}{2}} + C.$$

$y(-4)=0$ (ie $x=-4, y=0$)

$$0 = \frac{1}{3}(16+9)^{3/2} + C$$

So, $C = -\dfrac{125}{3}$

$$y = \frac{1}{3}(x^2+9)^{\frac{3}{2}} - \frac{125}{3}.$$

15) $a(t) = 4(t+3)^2$, $v_0 = -1$, $x_0 = 1$

$$a(t) = \frac{dv}{dt} = 4(t+3)^2$$

Integrating,

$$V(t) = \frac{4}{3}(t+3)^3 + C_1$$

$t=0$, $V_0 = -1$ gives

$$-1 = \frac{4}{3}3^3 + C_1 \Rightarrow C_1 = -37$$

So, $V(t) = \dfrac{dx}{dt} = \dfrac{4}{3}(t+3)^3 - 37$

Integrating,

$$x(t) = \frac{\cancel{4}}{3\times\cancel{4}}(t+3)^4 - 37t + C_2$$

$x_0 = 1$ ($t=0$, $x=0$)

$$1 = \frac{1}{3}3^4 + C_2 \Rightarrow C_2 = 1-27 = -26$$

Hence $x(t) = \dfrac{1}{3}(t+3)^4 - 37t - 26.$

18) $a(t) = 50\sin 5t$, $v_0 = -10$, $x_0 = 8$.

Here $a(t) = \dfrac{dV}{dt} = 50\sin 5t$

Integrating,

$$V(t) = -\frac{\overset{10}{\cancel{50}}}{\underset{1}{\cancel{5}}}\cos 5t + C_1$$

Using $V(0) = -10$

$$-10 = -10 + C_1 \Rightarrow C_1 = 0.$$

So, $V(t) = \dfrac{dx}{dt} = -10\cos 5t$

Integrating again,

$$x(t) = \frac{\overset{-2}{\cancel{-10}}}{\underset{1}{\cancel{5}}}\sin 5t + C_2$$

Using $x(0) = 8$,

$$8 = -2(0) + C_2 \Rightarrow C_2 = 8.$$

Hence,

$$x(t) = -2\sin 5t + 8.$$

27 (Additional) $x=0$, $t=0$, V_0 ——— $x=75$, $V=0$

At $x=0$, brakes are applied, $V_0 = ?$. At $x=75$, $V=0$.

$$a = \frac{dV}{dt} = -20 \Rightarrow$$

$$V(t) = -20t + C_1 \quad \text{———} \; *$$

$$t=0 \Rightarrow \boxed{C_1 = V_0}$$

$$V(t) = -20t + V_0$$

when $V=0$, $0 = -20t + V_0$

$$\Rightarrow \boxed{V_0 = 20t} \; **$$

From $*$ $V(t) = -20t + V_0$

So $x = -10t^2 + V_0 t + C_2$

Using $*$ $x = -10t^2 + (20t)t + C_2$

when $t=0$, $x=0$, so $\boxed{C_2 = 0}$

$$x = +10t^2$$

For $x = 75$, $t = \sqrt{\dfrac{75}{10}}$.

Hence $V_0 = 20t = 20\sqrt{\dfrac{75}{10}} = 10\sqrt{30}$