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DIFFRACTION OF SH-WAVES ACROSS A MIXED BOUNDARY IN A PLATE

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Abstract

The diffraction of SH-waves in an infinite elastic plate is studied under general boundary conditions using the Wiener-Hopf technique A mixed interface boundary value problem is solved as an illustartion.

1 INTRODUCTION

The problem of propagation of elastic waves in an elastic plate under a variety of boundary conditions has been studied extensively by many authors (see Achenbach [1] for details). One of the interesting problems in material sciences and non-destructive testing is that of determination of the diffracted field in the presence of inhomogeneities and discontinuities inherent in the elastic plate. Whenever the diffracting object can be modelled as a plane, the method proposed by Wiener and Hopf [9] to solve singular integral equations has been used by a number of authors. de Hoop [3] carried out an elegant and extensive study of the diffraction of plane waves in an infinite medium with a semi-infinite plane inclusion using the Wiener-Hopf method in the integral equation formulation. In this method the boundary value problem is reduced to an integral equation and then converted into the functional equation of the Wiener-Hopf type. Jones [4] presented a modification to this procedure by reducing the boundary value problem directly to the Wiener-Hopf equation without first having to derive an integral equation formulation. Since then a number of authors have applied this technique to obtain solution of a number of diffraction problems. Among them Kazi [5] ,Sinha [8] , Asghar and Zaman [2] and Zaman et. al. [10, 11] applied this to the diffraction problems arising from incident SH-waves on a plane discontinuity in an elastic medium.

The diffraction problems solved in the above cases involved either the displacement field (Neumann boundary conditions) or it's derivative (Dirichlet boundary conditions) to be specified on a half plane. In both the cases, the resulting Wiener-Hopf equation is solved to obtain the diffracted field. We use the general mixed boundary condition to be satisfied on a surface of an infinite elastic plate of uniform thickness. The lower surface of the plate is assumed to be free. The field in the plate is calculated using the Jones modification of the Wiener-Hopf technique. The problem of either displacement or the stress specified at a diffracting half plane can be derived as special cases from our case. An example of practical interest is presented to demonstrate usefulness of our model.

2 FORMULATION OF THE PROBLEM

Let us consider an infinite elastic plate of uniform thickness h. The axes are chosen such that the lower surface of the plate coincides with the plane $y = 0$. An SH-wave is assumed to be propagating in the positive direction of x-axis. The equation of motion is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\beta^2}\frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where $\beta = \sqrt{\frac{\mu}{\rho}}$ is the velocity of the shear wave, μ, ρ being the rigidity and density respectively of the medium. The zero initial conditions are assumed without any loss of generality. The boundary conditions are as follows.

1. On the lower surface of the plate the free surface boundary conditions are satisfied, i.e.,

$$\frac{\partial w}{\partial y} = 0 \quad on\ y = 0, \quad -\infty < x < \infty. \tag{2}$$

2. The upper surface of the plate satisfies mixed boundary conditions.

(a)

$$w(x, h, t) = f(x, h, t), \quad -\infty < x < 0, \tag{3}$$

(b)

$$\frac{\partial w(x, h, t)}{\partial y} = g(x, h, t), \quad 0 < x < \infty. \tag{4}$$

3. The radiation condition at infinity is assumed to ensure the uniqueness of the solution. (Noble [6]).

In addition, the functions $f(x, y, t)$ and $g(x, y, t)$ satisfy the conditions $f(x, y, t) = O(e^{pt})$ and $g(x, y, t) = O(e^{qt})$ for large t for some constants p, q. Also, $f(x, y, t)$ and $g(x, y, t)$ are assumed to be of exponential order in x as $x \to \infty$ or $x \to -\infty$ respectively for some real constants A, B, τ_1 and τ_2.

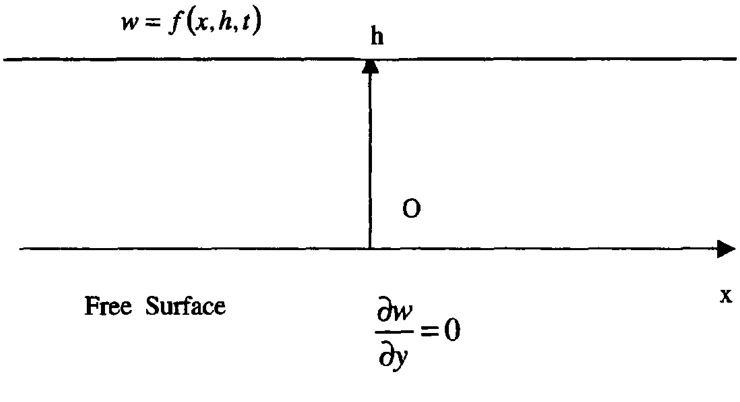

Figure 1: Geometery of the problem

3 WIENER-HOPF EQUATION

Let us define the Laplace transform in t as

$$\mathcal{L}\{w(t)\} = W(s) = \int_0^\infty w(t)e^{-st}dt, \tag{5}$$

where s is a complex parameter such that $p, q < \text{Re}(s)$. The Fourier transform in x is defined as

$$\mathcal{F}\{w(x)\} = w^*(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx, \tag{6}$$

where $\alpha = \sigma + i\tau$. The half range Fourier transforms are defined as

$$w_+^*(\alpha) = \frac{1}{2\pi}\int_0^\infty f(x)e^{i\alpha x}dx \tag{7}$$

and

$$w_-^*(\alpha) = \frac{1}{2\pi}\int_{-\infty}^0 f(x)e^{i\alpha x}dx, \tag{8}$$

so that

$$w^*(\alpha) = w_+^*(\alpha) + w_-^*(\alpha). \tag{9}$$

If $|w(x)| < Ae^{\tau_1 x}$ as $x \to -\infty$, $w_-^*(\alpha)$ defines an analytic function of α if $\text{Im}(\alpha) = \tau < \tau_1$. Similarly, if $|w(x)| < Be^{\tau_2 x}$ as $x \to \infty, w_+^*(\alpha)$ is analytic if $\tau > \tau_2$. Thus $w^*(\alpha)$ defines an analytic function of α in the common strip $\tau_1 < \mathrm{Im}(\alpha) = \tau < \tau_2$.

We now take the Laplace transform in t and the Fourier transform in x of the equation (1) to obtain

$$\frac{d^2W^*(x,\alpha,s)}{dy^2} - \gamma^2 W^*(x,\alpha,s) = 0, \tag{10}$$

where $\gamma^2 = \alpha^2 + \frac{s^2}{\beta^2}$. The transformed boundary conditions can be written as

$$\frac{dW^*}{dy} = 0; \quad y = 0, -\infty < x < \infty. \tag{11}$$

$$W_-^*(\alpha,h,s) = F_-^*(\alpha,h,s); \quad y = h, -\infty < x < 0. \tag{12}$$

$$W_+^{*/}(\alpha,h,s) = G_+^*(\alpha,h,s); \quad y = h, 0 < x < \infty. \tag{13}$$

For brevity sake we shall suppress the dependence on some or all of α, y, s and will write these only when needed. The solution to equation (11)can be written as

$$W^*(\alpha,y) = C_1 e^{\gamma y} + C_2 e^{-\gamma y}, \quad 0 < y < h. \tag{14}$$

Using the boundary conditions (12), we get

$$W^*(\alpha,y) = A(\alpha,s)\cosh\gamma y, \tag{15}$$

where $C_1 = C_2 = A(\alpha,s)$. Using the decomposition formula (10), we obtain from (15)

$$W_+^*(\alpha,h) + W_-^*(\alpha,h) = A(\alpha,s)\cosh\gamma h, \tag{16}$$

$$W_+^{*/}(\alpha,h) + W_-^{*/}(\alpha,h) = -\gamma A(\alpha,s)\sinh\gamma h. \tag{17}$$

We eliminate $A(\alpha,s)$ from equations (16) and (17) using the boundary conditions (13) and (14) to get the Wiener-Hopf equation

$$G_+^*(\alpha,h) - W_-^{*/}(\alpha,h) = -\gamma\tanh\gamma h\left\{W_+^*(\alpha,h) + F_-^*(\alpha,s)\right\} \tag{18}$$

4 SOLUTION OF THE WIENER-HOPF EQUATION

Sato [7] has given the factorization of $\frac{\sinh \gamma h}{\gamma h}$ and $\cosh \gamma h$ in the one sided functions which are analytic either in the upper or lower half-plane. Without going into details, we give Sato's results in Appendix I and use these to obtain factorization

$$\gamma \tanh \gamma h = \frac{P_+(\alpha)}{P_-(\alpha)}, \tag{19}$$

where $P_\pm(\alpha)$ are given by equation (A_7) in the appendix I. The Wiener-Hopf equation can thus be written as

$$P_-(\alpha)G_+^*(\alpha) - P_-(\alpha)W_-^{*\prime}(\alpha,h) = P_+(\alpha)W_+^*(\alpha,h) + P_+(\alpha)F_-^*(\alpha). \tag{20}$$

We split the mixed terms still present in equation (20) into the sum of one sided functions using the general decomposition theorem given by Noble [6] as

$$M(\alpha) = P_+(\alpha)F_-^*(\alpha) - P_-(\alpha)G_+^*(\alpha) = M_+(\alpha) + M_-(\alpha). \tag{21}$$

The expressions for $M_+(\alpha)$ and $M_-(\alpha)$ are given as

$$M_\pm(\alpha) = \pm\frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{M(\zeta)}{\zeta-\alpha}d\zeta. \tag{22}$$

The equation (20) can thus be written as

$$M_-(\alpha) - P_-(\alpha)W_-^{*\prime}(\alpha,h) = P_+(\alpha)W_+^*(\alpha,h) - M_+(\alpha). \tag{23}$$

The left side of equation (23) is analytic in the lower half-plane $\tau < \tau_2$ while the right side is analytic in the upper half-plane $\tau > \tau_1$. Both sides are equal in the strip $\tau_1 < \tau < \tau_2$ and thus define an entire function. By Liouville's theorem and appropriate asymptotic behaviour (Noble[6]), this entire function can be shown to be zero. Thus

$$W_-^{*\prime}(\alpha,h,s) = \frac{M_-(\alpha)}{P_-(\alpha)} \tag{24}$$

and

$$W_+^*(\alpha,h,s) = \frac{M_+(\alpha)}{P_+(\alpha)}. \tag{25}$$

We note that $P_{\pm}(\alpha)$ are given through equation (A_7) in terms of the transformed boundary data $F_{-}^{*}(\alpha,s)$ and $G_{+}^{*}(\alpha,s)$ while $M_{\pm}(\alpha)$ are give by equation (22) in terms of known functions. The equations (24) and (25) can be used in conjunction with (13), (14) and (15) to determine the displacement field in the transformed plane. The inverse Laplace transform and the Fourier inversion formula can then be used to obtain the displacement $w(x,y,t)$. In the following section, we exhibit it in a case of practical interest.

4.1 FREE-RIGID BOUNDARY

We consider the case in which semi-infinite half plane which forms the right half of the upper surface of the plate is stress free while the semi-infinite half-plane forming the rest of the upper surface of the plate is assumed to be rigid. A time harmonic SH wave is travelling in the negative direction of the x-axis so that the displacement field is given by

$$w(x,y) = C\cosh q(y-h)e^{-\imath kx}, \tag{26}$$

where C is a constant, $q = \frac{\varpi^2}{\beta^2} - k^2, \varpi$ is angular frequency, β is shear velocity and k is the wave number. The time dependence factor $e^{\imath\varpi t}$ is omitted for brevity. At $y = h$, $x > 0$, the boundary condition is given by

$$\frac{\partial w(x,h)}{\partial y} = g(x) = 0, \quad 0 < x < \infty. \tag{27}$$

The total displacement vanishes at the rigid boundary $x < 0$ imposing the condition

$$w(x,h) = f(x) = -Ce^{-\imath kx}; \quad -\infty < x < 0, \tag{28}$$

In this case we have

$$\begin{aligned} F_{-}^{*}(\alpha) &= \frac{-C}{\imath(\alpha-k)}, \\ G_{+}^{*}(\alpha) &= 0 \end{aligned} \tag{29}$$

The Wiener-Hopf equation (21) in this case becomes

$$-P_{-}(\alpha)w_{-}^{*\prime}(\alpha,h) = P_{+}(\alpha)w_{+}^{*}(\alpha,h) - \frac{P_{+}(\alpha)C}{\imath(\alpha-k)}. \tag{30}$$

It follows that the displacement in the transformed domain can be written as

$$w^{*}(\alpha,y) = \frac{CP_{+}(k)}{(\alpha-k)P_{-}(\alpha)\gamma\sinh\gamma h}\cosh\gamma y. \tag{31}$$

Thus, taking inverse Fourier transform we get

$$w(x,y) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{AP_+(k)\cosh\gamma y}{(\alpha-k)P_-(\alpha)\gamma\sinh\gamma h} e^{-i\alpha x} d\alpha. \tag{32}$$

Notice that the pole $\alpha = k$ gives rise to a wave that exactly cancels with the incident wave as should be expected. Other poles are given by $\sin\gamma h = 0$, giving $\gamma = \frac{in\pi}{h}$. This gives $\alpha^2 - \frac{n^2\pi^2}{h^2} = -p_n^2$ (say). Enclosing the contour in the upper half plane, the contributions of the poles $\alpha = ip_n$ give the displacement field

$$w(x,y) = \sum_{n=1}^{\infty} \frac{AP_+(k)\cosh\frac{in\pi}{h}y}{(p_n - ik)P_-(ip_n)\frac{d}{d\alpha}\left[\sinh\gamma h\right]_{\alpha=ip_n}} e^{p_n x}. \tag{33}$$

This gives the transmitted field in the left part of the plate. A similar analysis can be carried out if a wave is incident on the part of the plate with free-free boundary from the part of the plate with free-rigid boundary.

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6 APPENDIX I

1. Sato [6] has described the factorization of $\frac{\sin\gamma h}{\gamma h}$ by writing it as

$$\frac{\sin\gamma h}{\gamma h} = \prod_{n=1}^{\infty}\{q_n^2 h_n + \alpha^2 h_n^2\} = H(\alpha), \tag{A_1}$$

where $\gamma = (\alpha^2 + \frac{s^2}{\beta^2})$, $q_n h_n = (1 - \frac{s^2}{\beta^2}h_n^2)^{1/2}$, $h_n = \frac{h}{n\pi}$.
So that $H(\alpha) = H_+(\alpha)H_-(\alpha)$,
where

$$\begin{aligned} H_\pm(\alpha) &= \prod_{n=1}^{\infty}(q_n h_n \mp i\alpha h_n)\exp\{\mp i\alpha h_n \pm \chi(\alpha)\}, \qquad (A_2)\\ \chi(\alpha) &= -(i\alpha h/\pi)\left\{1 - C - \log\frac{\alpha h}{\pi}\right\} + \frac{\alpha h}{2}, \end{aligned}$$

C being the Euler constant. Thus

$$\frac{\sin\gamma h}{\gamma h} = H_+(\alpha)H_-(\alpha). \tag{A_3}$$

2. The infinite product representation of $\cosh\gamma h$ is

$$\cosh\gamma h = \prod_{n=1}^{\infty}\left\{1 + \frac{4\gamma^2 h^2}{(2n-1)^2\pi^2}\right\} = \prod_{n=1}^{\infty}\left\{q_n^2\bar{h}_n + \alpha^2\bar{h}_n^2\right\} \tag{A_4}$$

where $\bar{h}_n = \frac{h}{(n-1/2)\pi}$, $q_n\bar{h}_n = (1 - \frac{s^2}{\beta^2}\bar{h}_n^2)^{1/2}$ So that we can write

$$\cosh\gamma h = L_+(\alpha)L_-(\alpha), \tag{A_5}$$

where

$$L_\pm(\alpha) = \prod_{n=1}^{\infty}(q_n\bar{h}_n \mp i\alpha\bar{h}_n)\exp\left\{\mp i\alpha\bar{h}_n \pm \chi(\alpha)\right\}. \tag{A_6}$$

3. We can now write the function

$$\begin{aligned} \gamma\tanh\gamma h &= (\alpha^2 + \frac{s^2}{\beta^2})h\frac{\sinh\gamma h}{\gamma h\cosh\gamma h}\\ &= (\alpha + i\frac{s}{\beta})(\alpha - i\frac{s}{\beta})h\frac{H_+(\alpha)H_-(\alpha)}{L_+(\alpha)L_-(\alpha)}\\ &= h\frac{(\alpha + i\frac{s}{\beta})H_+(\alpha)}{L_+(\alpha)}\Big/\frac{L_-(\alpha)H_-(\alpha)}{(\alpha - i\frac{s}{\beta})}\\ &= \frac{P_+(\alpha)}{P_-(\alpha)}. \qquad (A_7) \end{aligned}$$