Asymptotic Solutions of Differential Eqns

We shall consider linear second order homogeneous DE for $w(z)$ of the form

$$w''(z) + p\,w'(z) + q\,w(z) = 0 \qquad \text{—— (1)}$$

where z may be a complex number, p, q are analytic except possibly at ∞. These may depend upon z and some parameter. We wish to find $w(z)$ as $z \to \infty$ or as some parameter $\to \infty$. We shall not adopt the method of first finding the solution and then finding asymptotic form of the solution.

We put $w(z) = W(z)\, e^{-\frac{1}{2}\int p(z)dz}$

which reduces (1) to

$$W'' + \left(q - \tfrac{1}{2}p' - \tfrac{1}{4}p^2\right) W = 0$$

Thus we may consider

$$W'' + f(z)\, W = 0 \qquad \text{—— (2)}$$

$f(z)$ is related to p and q.

We shall consider two asymptotic problems

(i) $z \to \infty$

(ii) an independent parameter $\lambda \to \infty$.

(ii) falls under a type of singular perturbation theory

Ordinary and Singular points

If $z = \infty$ is an ordinary point then as $z \to \infty$, $W(z)$ consists of two independent series expansions, in inverse powers of z which are convergent for $|z| > R$ for some R. If the point at ∞ is not an ordinary point, it is a singularity.

The point $z=\infty$ is a regular singularity of $w(z)$
if $f(z) = O(z^{-2})$ as $z \to \infty$
(In terms of p, q it means that $p = O(z^{-1})$ and $q = O(z^{-2})$)
In fact we require $z^2 f$, zp, $z^2 q$ to be analytic as $z \to \infty$.
In this case the usual Frobenius method

$$w(z) = z^r \sum_{n=0}^{\infty} c_n z^{-n}$$ is used.

The point $z=\infty$ is an irregular singularity of $w(z)$ if $f(z)$ is not $O(z^{-2})$ as $z \to \infty$.
Examples are $f(z) = O(1)$, $f(z) = O(z)$ or $O(z^{-1})$ as $z \to \infty$.
Equations with irregular singular point are discussed here.

<u>Motivation</u>: Let us consider

$$f(z) \sim a_0 + \frac{a_1}{z^2} + \cdots = a_0 + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty$$

where $a_0 \neq 0$. A term $O(z^{-1})$ is omitted for simplicity. Such a term will be included in the general methods.

Asymptotic expansion for $w(z)$ is expected as

$$w(z) \sim w_0(z) + w_1(z) + \cdots \qquad (3)$$

where $w_n = o(w_{n-1})$

Put (3) in (2), we get

$$(w_0'' + w_1'' + \cdots) + \left(a_0 + \frac{a_2}{z^2} + \cdots\right)(w_0 + w_1 + \cdots) \sim 0 \qquad (4)$$

The dominant term then satisfies

$$w_0'' + a_0 w_0 = 0 \implies w_0(z) = A_0 e^{i a_0^{1/2} z} + B_0 e^{-i a_0^{1/2} z}$$

(4) then gives

$$w_1'' + a_0 w_1 = -\frac{a_2}{z^2} w_0 = O\left(z^{-2} e^{\pm i a_0^{1/2} z}\right) \quad \text{— (5)}$$

Since we require $w_1 = o(w_0)$, we need only the particular integral of (5) which is

$$w_1(z) = O\left(z^{-1} e^{\pm i a_0^{1/2} z}\right) \quad \text{— (6)}$$

Thus (4) and (6) together give us

$$w(z) = e^{i a_0^{1/2} z}\left\{A + O(z^{-1})\right\} + e^{-i a_0^{1/2} z}\left\{B + O(z^{-1})\right\}$$

where A, B are constants. — (7)

<u>General Method</u>.

Let us now start with a more general form

$$f(z) \sim a_0 + \frac{a_1}{z} + \cdots \quad \text{as } z \to \infty \quad \text{— (8)}$$

and consider

$$\left.\begin{aligned} w(z) &= e^{\lambda z} z^{\sigma} g(z) \\ g(z) &= \alpha_0 + \frac{\alpha_1}{z} + \cdots \end{aligned}\right\} \text{as } z \to \infty \quad \text{(as above considerations suggests!)} \quad (9)$$

where a_n, $n = 0, 1, 2, \ldots$ are constants and are obtained from (8) and α_n, $n = 0, 1, 2, \ldots$, λ and σ have to be found.

We put (8) and (9) in (2), we obtain by successive approximation two asymptotic solutions for $w(z)$ as $z \to \infty$. From (9)

$$w''(z) = e^{\lambda z} z^{\sigma}\left\{\alpha_0 \lambda^2 + \frac{1}{z}(\lambda^2 \alpha_1 + 2\lambda\sigma\alpha_0) + \cdots\right\}$$

Put this in D.E (2), after cancelling $e^{\lambda z}$ and z^{a}, collection of terms of like powers in z gives

$$\left\{\alpha_0(\lambda^2 + a_0)\right\} + \frac{1}{z}\left\{\alpha_1(\lambda^2 + a_0) + \alpha_0(2\lambda\sigma + a_1)\right\} + \cdots = 0$$

Successively applying the limit $z \to \infty$, which is equivalent to setting coeff of each power of z equal to zero, we get

$$\lambda^{(1)}, \lambda^{(2)} = \pm i a_0^{1/2}$$

$$\sigma_1^{(1)}, \sigma^{(2)} = -\frac{a_1}{2\lambda} = \pm \frac{i a_1}{2 a_0^{1/2}}$$

$$\alpha_1^{(1)}, \alpha_1^{(2)} = \mp \frac{i \alpha_0}{2 a_0^{1/2}} \left(\mp \frac{i a_1}{2 a_0^{1/2}} - \frac{a_1^2}{4 a_0} + a_2 \right),$$

$$\alpha_{n+1} = \frac{1}{2\lambda(n+1)} \Big[\{ n(n+1) - (2n+1)\sigma + \sigma^2 + a_2 \} \alpha_n + a_3 \alpha_{n-1} + \dots + a_{n+2} \alpha_0 \Big], \quad n = 0, 1, 2, \dots$$

where

$\alpha_{-k} = 0$ for $k = 1, 2, \dots$, α_0 is arbitrary.

~~Example~~ $\cancel{w'' + \frac{1}{z} w = 0}$, $\cancel{z \to \infty}$

~~Let us assume~~ $\cancel{w(z) = e^{\lambda z} z^{\sigma} g(z)}$

$$\cancel{= e^{\lambda z} z^{\sigma} \left(\alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \dots \right)} \quad \text{as } z \to \infty$$

~~Here~~ $\cancel{f(z) \sim a_0 + \frac{a_1}{z} + \dots}$ ~~with~~ $\cancel{a_0 = a_1 = a_2 = 0}$

$$w_1(z) \sim A e^{\lambda^{(1)} z} z^{\sigma^{(1)}} \left(1 + \frac{\alpha_1^{(1)}}{\alpha_0} \cdot \frac{1}{z} + \dots \frac{\alpha_n^{(1)}}{\alpha_0} \frac{1}{z^n} + \dots \right)$$

$$w_2(z) \sim B e^{\lambda^{(2)} z} z^{\sigma^{(2)}} \left(1 + \frac{\alpha_1^{(2)}}{\alpha_0} \cdot \frac{1}{z} + \dots \frac{\alpha_n^{(2)}}{\alpha_0} \frac{1}{z^n} + \dots \right)$$

Meth 2 continued

Suppose now $f(z)$ is such that $a_0 = 0$ (the above procedure demands $a_0 \neq 0$). Suppose $a_1 \neq 0$. In this case the D.E. may be of the form

$$w'' + \frac{1}{z} F(z) w = 0 \qquad \text{——— (10)}$$

$$F(z) \sim a_1 + \frac{a_2}{z} + \cdots \quad \text{as } z \to \infty$$

Put $z = Z^2$, $w = Z^{1/2} W(Z)$, DE (10) becomes

$$W''(Z) + G(Z) W(Z) = 0 \qquad \text{——— (11)}$$

$$G(Z) = 4F(Z^2) - \frac{3}{4Z^2} \sim 4a_1 + O\left(\frac{1}{Z^2}\right)$$

which is now in the same form as $f(z)$ in the first case. $\left(f(z) \sim a_0 + O\left(\frac{1}{z^2}\right)\right)$

As before (instruction) $W(Z) = C e^{\pm i (4a_1)^{1/2} Z} \left\{ 1 + O\left(\frac{1}{Z}\right) \right\}$ — (12)

and so $w(z) = C z^{1/4} e^{\pm 2 i (a_1)^{1/2} z^{1/2}} \left\{ 1 + O \frac{1}{z^{1/2}} \right\}$ — (13)

~~(The case $a_0 = 0$, $a_1 = 0$, $a_2 \neq 0$ is regular singular point)~~

Calculations

$$w = Z^{1/2} W(Z) \qquad z = Z^2, \quad \frac{dz}{dZ} = 2Z$$

$$w' = \frac{dw}{dZ} \frac{dZ}{dz}$$

$$= \frac{1}{2Z}\left[\frac{1}{2} Z^{-1/2} W(Z) + Z^{1/2} W'(Z)\right]$$

$$= \frac{1}{4} Z^{-3/2} W(Z) + \frac{1}{2} Z^{-1/2} W'(Z)$$

$$-5-1, \quad -\tfrac{7}{2} + \tfrac{3}{2}$$

$$w'' = \frac{1}{2Z}\left[\frac{1}{4}\left(-\frac{3}{2}\right) Z^{-5/2} W(Z) + \cancel{\frac{1}{4} Z^{-3/2} W'(Z)} - \cancel{\frac{1}{4} Z^{-3/2} W'(Z)} + \frac{1}{2} Z^{-1/2} W''(Z)\right]$$

$$= -\frac{3}{16} Z^{-7/2} W(Z) + \frac{1}{4} Z^{-3/2} W''(Z) \qquad Z^{-3/2}$$

$$w'' + \frac{1}{z} F(z)\, w = -\frac{3}{16} Z^{-7/2} W(Z) + \frac{1}{4} Z^{-3/2} W''(Z) + \frac{1}{Z^2}\left[Z^{1/2} W(Z)\right]$$

$$= -\frac{3}{4} Z^{-2} W(Z) + W''(Z) + 4 W(Z) \checkmark$$

Example. $w'' - \frac{1}{z} w = 0$

$$f(z) = \frac{1}{z} \qquad (a_0 = 0 \;, a_1 = 1, a_2 \cdots = 0)$$

$$-\frac{7}{2} + \frac{3}{2}$$

put $z = Z^2$, $w = Z^{1/2} W(Z)$

$$w'' = -\frac{3}{16} Z^{-7/2} W(Z) + \frac{1}{4} Z^{-3/2} W''(Z)$$

D.E $\Rightarrow$

$$-\frac{3}{16} Z^{-7/2} W(Z) + \frac{1}{4} Z^{-3/2} W''(Z) - \underline{\underline{Z^{-2} Z^{1/2}}} W(Z) = 0$$

This gives $W''(Z) + W(Z)\left[-\frac{3}{4} Z^{-2} + 1\right] = 0$

$$f(Z) = 1 - \frac{3}{4Z^2} \qquad (\text{motivation case applies})$$

$$a_1 = -\frac{3}{4}.$$

The solution can be written.

Method 2:

Motivation: the supposition

$W(z) = e^{\lambda z} z^{\sigma} g(z)$ can be written

as
$$W(z) = e^{\lambda z + \log z^{\sigma} + \log(g(z))}$$

$$\sim e^{\lambda z + \sigma \log z + \log(\alpha_0 + \frac{a_1}{z} + \cdots)} \quad \text{as } z \to \infty$$

$$= e^{\lambda z + \sigma \log z + \log \alpha_0 + \log(1 + \frac{\alpha_1}{\alpha_0 z} + \cdots)}$$

$$= \alpha_0 e^{\lambda z + \sigma \log z + \frac{\alpha_1}{\alpha_0} z^{-1} + O(z^{-2})} \quad \text{as } z \to \infty$$

The exponent of exponential

$$\underline{\lambda z} + \underline{\sigma \log z} + \frac{\alpha_1}{\underline{\alpha_0}} \frac{1}{z} + O\left(\frac{1}{z^2}\right)$$

can be seen to be asymptotic as $z \to \infty$

This suggests that we may try to use an asymptotic series in the exponent of the exponential.

Method. Assume

$$w(z) \sim e^{\phi_0(z) + \phi_1(z) + \cdots} \quad \text{as } z \to \infty \qquad (14)$$

where $\{\phi_n(z)\}$, $n = 0, 1, 2..$ is an asymptotic sequence as $z \to \infty$. If ϕ_n can be differentiated twice $\{\phi_n''\}$ and $\{\phi_n'\}$ will also be asymptotic. We shall proceed on this basis. Using

$$w'' + f(z)\, w = 0 \qquad (15)$$

as a starting point, we put the above eqn (14) in it to get

$$\phi_0'' + \phi_1'' + \cdots + (\phi_0' + \phi_1' + \cdots)^2 + a_0 + \frac{a_1}{z} + \cdots \sim 0 \qquad (16)$$

which determines ϕ_n, $n = 0, 1, ..$ by successively applying the asymptotic limit $z \to \infty$.

The leading term in (16), comes from $f(z)$ which is a_0 here. a_0 is $O(1)$ term. The leading terms from $(\phi_0'' + \phi_1'' + \cdots) + (\phi_0' + \phi_1' + \cdots)^2$ must cancel this a_0: this determines ϕ_0. Since ϕ_0 occurs as derivatives: ϕ_0'' and $\phi_0'^2$, we must have ϕ_0 equal to some positive power of z (why!). This implies that asymptotically $\phi_0'' = o(\phi_0'^2)$ which in turn implies that the leading term must be $\phi_0'^2$. (16) thus gives as $O(1)$ term [DE7]

$$\phi_0'^2 + a_0 \sim 0$$

$$\Rightarrow \quad \phi_0(z) = \lambda z + O(1), \sim \lambda z, \quad \lambda = \pm i a_0^{1/2} \qquad (17)$$

We can now proceed to determine higher order terms. From (17), $\phi_0'' \equiv 0$, (16) gives by equating the next asymptotic terms to ($O(z^{-1})$) to zero,

$$2\phi_0'\phi_1' + \frac{a_1}{z} \sim 0$$

Putting value of $\phi_0(z)$ from (17)

$$\phi_1' = -\frac{a_1}{2\lambda} z \Rightarrow \phi_1(z) = \sigma \log z + O(1)$$
$$\sim \sigma \log z \qquad (18)$$

where $\sigma = -\dfrac{a_1}{2\lambda}$

Using (17) and (18) in (16), $O(z^{-2})$ gives

$$\phi_1'' + \phi_1'^2 + 2\phi_0'\phi_2' + \frac{a_2}{z^2} \sim 0$$

which leads to

$$\phi_2(z) \sim \frac{1}{z}\,\frac{1}{2\lambda}\left(\frac{a_1}{2\lambda} + \frac{a_1^2}{4\lambda^2} + a_2\right) \qquad (19)$$

Thus $O(z^{-n})$ terms will give $\phi_n(z)$, $n \geq 3$ and $\phi_n(z)$ will be $O(z^{-n+1})$

Putting all these values in (14), grouping all constants of integration into one constant $\log \alpha_0$ (for convenience)

$$w(z) = e^{\lambda z + \sigma \log z + \log \alpha_0 + \phi_2(z) + \cdots}$$

$$= \alpha_0 z^{\sigma} e^{\lambda z} \{ 1 + \phi_2(z) + \cdots \}$$

$$= e^{\lambda z} z^{\sigma} \left\{ \alpha_0 + \frac{1}{z} \frac{\alpha_0}{2\lambda} \left(\frac{a_1}{2\lambda} + \frac{a_1^2}{4\lambda^2} + a_2 \right) + O\left(\frac{1}{z^2}\right) \right\} \qquad (20)$$

where λ turns out to be the same $(= \pm i a_0^{1/2})$ as in Method #1.

<u>Example</u> Consider the so-called parabolic cylinder equation (solutions are called parabolic cylinder functions)

$$w'' - z^2 w = 0$$

The method #1 does not seem to be suitable because of form of $f(z)$.

Putting $w(z) \sim e^{\phi_0(z) + \phi_1(z) + \cdots}$ as $z \to \infty$

we get
$$\phi_0'' + \phi_1'' + \cdots + \phi_0'^2 + 2\phi_0'\phi_1' + \phi_1'^2 + 2\phi_0'\phi_2' + \cdots - z^2 \sim 0$$

Since the dominant term is $-z^2$, we must again conclude that ϕ_0 equal to some positive power Az, which implies $\phi_0'' = o(\phi_0'^2)$ which must contribut a term $O(z^2)$ to cancel $-z^2$.

We must try $\phi_0 = O(z^2)$

$$\phi_0'' = O(1) = o(\phi_0'^2).$$

The dominant term gives $\phi_0'^2 - z^2 \sim 0 \Rightarrow \phi_0(z) \sim \pm z^2/2$ as $z \to \infty$

This gives $\phi_0'' \sim \pm 1$ which will be taken into account in $O(1)$ terms.

The second order terms here ~~at~~ are $O(z)$, these give

$$\phi_1'' + 2(\pm z)\phi_1' + \phi_1'^2 \sim 0$$

$$\Rightarrow \phi_1(z) \equiv 0 \quad \text{as the appropriate solution}$$

$O(1)$ terms $\Rightarrow$

$$\phi_0'' + 2\phi_0'\phi_2' \sim 0 \Rightarrow (\pm 1) + 2(\pm z)\phi_2' \sim 0$$

$$\Rightarrow \phi_2(z) \sim -\frac{1}{2}\log z$$

$$O(z^{-1}) \Rightarrow \quad 2\phi_0'\phi_3' \sim 0 \Rightarrow \phi_3(z) \equiv 0$$

$$O(z^{-2}) \Rightarrow \quad \phi_2'' + \phi_2'^2 + 2\phi_0'\phi_4' \sim 0$$

$$\Rightarrow \phi_4(z) \sim \pm\frac{3}{16}\frac{1}{z^2}$$

Thus

$$\left.\begin{aligned} w_1(z) &\sim A\, e^{z^2/2} z^{-1/2}\left\{1 + \frac{3}{16}\frac{1}{z^2} + O\left(\frac{1}{z^4}\right)\right\} \\ w_2(z) &\sim B\, e^{-z^2/2} z^{-1/2}\left\{1 - \frac{3}{16}\frac{1}{z^2} + O\left(\frac{1}{z^4}\right)\right\} \end{aligned}\right] \text{as } z \to \infty$$

A, B are arbitrary constants

References

Perturbation Theory : Lectures in Applied Math (vol 16) 1977)

Kevorkian, Cole. 1981) Perturbation methods in applied mathematics

Asymptotic Solutions of DE's (DE9a)

Ex Consider the Bessel equation of order 1/2

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0 \quad \text{—} (1)$$

Put in standard form

$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0$$

$$P(x) = \frac{1}{x}$$

Put $y = u(x)\, e^{-\int \frac{1}{2} P(x)dx} = x^{-\frac{1}{2}} u(x)$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} u'(x) - \frac{1}{2} x^{-\frac{3}{2}} u(x)$$

$$\frac{d^2y}{dx^2} = x^{-\frac{1}{2}} u''(x) \underbrace{- \frac{1}{2} x^{-\frac{3}{2}} u'(x) - \frac{1}{2} x^{-\frac{3}{2}} u'(x)} + \frac{3}{4} x^{-\frac{5}{2}} u(x)$$

Put in (1)

$$x^2\left(x^{-1/2} u''(x) - x^{-3/2} u'(x) + \frac{3}{4} x^{-\frac{5}{2}} u(x)\right) + x\left(x^{-\frac{1}{2}} u'(x) - \frac{1}{2} x^{-\frac{3}{2}} u(x)\right)$$
$$+ \left(x^2 - \frac{1}{4}\right) x^{-\frac{1}{2}} u(x) = 0$$

collecting like terms

$$x^{3/2} u''(x) + \left[\cancel{-x^{1/2}} + \cancel{x^{1/2}}\right] u'(x) + \left[\cancel{\frac{3}{4} x^{-1/2}} - \cancel{\frac{1}{2} x^{-1/2}} + x^{3/2} - \cancel{\frac{1}{4} x^{-1/2}}\right] u(x) = 0$$

This simplifies to $u''(x) + u(x) = 0$

Thus for $f(x) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$

$$a_0 = 1, \quad a_1 = a_2 = \cdots = 0$$

To apply 1st method solution is $u(x) = e^{\lambda x} x^{\sigma}\left(\alpha_0 + \frac{\alpha_1}{z} + \cdots\right)$

where $\lambda = \pm i a_0^{1/2} = \pm i$

$$\sigma = 0 \quad \left(= \frac{-a_1}{2\lambda}\right)$$

$\alpha_1^{(1)}, \alpha_1^{(2)}$ are zero since a_1 is zero.

Thus $u(x) \sim e^{\pm ix} = \cos x \pm i \sin x \; (\equiv \cos x, \sin x)$

This gives $\boxed{y = x^{-1/2}(C_1 \cos x + C_2 \sin x)}$

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Ex. $x y'' + 2(1-x) y' - y = 0$, x large.

Put in the standard form

$$y'' + 2\left(-1 + \frac{1}{x}\right) y' - \frac{1}{x} y = 0$$

$$e^{-\frac{1}{2}\int p(x)\,dx} = e^{-\int\left(-1+\frac{1}{x}\right)dx} = x^{-1} e^{x}$$

Put $y = x^{-1} e^{x} Y(x)$

$$\frac{dy}{dx} = x^{-1} e^{x} Y' - x^{-2} e^{x} Y(x) + x^{-1} e^{x} Y(x)$$

$$\frac{d^2y}{dx^2} = x^{-1} e^{x} Y'' - x^{-2} e^{x} Y' + x^{-1} e^{x} Y'$$
$$- x^{-2} e^{x} Y' + 2 x^{-3} e^{x} Y - x^{-2} e^{x} Y$$
$$+ x^{-1} e^{x} Y' - x^{-2} e^{x} Y + x^{-1} e^{x} Y$$

Put in D.E. and cancel e^{x} throughout

$$x\left(x^{-1} Y'' + 2 x^{-1} Y' - 2 x^{-1} Y' - 2 x^{-1} Y + 2 x^{-2} Y + x^{-1} Y\right)$$
$$+ 2 x^{-1} Y' - 2 x^{-2} Y + 2 x^{-1} Y$$
$$- 2 e^{x} + 2 x^{-1} Y - 2 Y - x^{-1} Y = 0$$

simplification gives

$$Y'' + \left(-1 + \frac{1}{x}\right) Y(x) = 0$$

So $f(x) \sim a_0 + \frac{a_1}{x} + \cdots$ gives $a_0 = -1$, $a_1 = 1$, $a_n = 0, n \geq 2$

This gives

$\lambda = \pm i (-1)^{1/2} = \pm 1$

$\sigma = \frac{\pm i a_1}{2 a_0^{1/2}} = \pm \frac{i \cdot 4}{2i} = \pm 2$

$\alpha_1^{(1)}, \alpha_1^{(2)} = \cdots$

Finally use $y = x^{-1} e^{x} Y(x)$.