Chapter 6

Legendre Equation and Polynomials

In solving the wave and heat equation in spherical coordinates, we come across the Legendre equation

$$(1-x^2)\,y''(x) - 2x\,y'(x) + \lambda\,y(x) = 0 \qquad \text{— (1)}$$

or $$\left[(1-x^2)\,y'(x)\right]' + \lambda\,y(x) = 0 \quad , \quad -1 < x < 1.$$

As mentioned in chapter 5, at $x = \pm 1$, $r(x) = 1-x^2$ vanishes and so this gives rise to a singular Sturm-Liouville problem. In the ~~case~~ present case, no boundary condition is specified at the end points ± 1.

From power series method of solution point of view, $x=0$ is an ordinary point so we can assume

$$y(x) = \sum_{k=0}^{\infty} a_k x^k .$$

so

$$y'(x) = \sum_{k=1}^{\infty} a_k\, k\, x^{k-1} \;;\; y''(x) = \sum_{k=2}^{\infty} a_k\, k(k-1)\, x^{k-2}$$

Put in D.E (1), we get

$$(1-x^2)\sum_{k=2}^{\infty} a_k\, k(k-1)\, x^{k-2} - 2x\sum_{k=1}^{\infty} a_k\, k\, x^{k-1} + \lambda\sum_{k=0}^{\infty} a_k x^k = 0$$

or

$$-\sum_{k=2}^{\infty} a_k\, k(k-1)\, x^{k} + \sum_{k=2}^{\infty} a_k\, k(k-1)\, x^{k-2} - \sum_{k=1}^{\infty} 2a_k\, k\, x^k + \lambda\sum_{k=0}^{\infty} a_k x^k = 0$$

~~Now we make the indices on Σ the same by~~

or

$$-\sum_{k=2}^{\infty} a_k\, k(k-1)\, x^k + \underbrace{\sum_{k=0}^{\infty} a_{k+2}(k+1)(k+2)\, x^k}_{\substack{k-2=m \\ k=2 \Rightarrow m=0 \\ \text{Then change } m \to k}}$$

$$-\sum_{k=1}^{\infty} 2 a_k\, k\, x^k + \lambda \sum_{k=0}^{\infty} a_k x^k = 0$$

Now we take out terms from inside Σ's so all Σ's start from $k = 2$. Collect the terms to write

$$2a_2 + 6a_3 x + \lambda a_0 + \lambda a_1 x - 2a_1 x + \sum_{k=2}^{\infty}\left[(k+1)(k+2)\, a_{k+2} - \{k(k-1) + 2k - \lambda\} a_k\right] x^k = 0$$

comparing coefficients of like powers of x,

$$2a_2 = -\lambda a_0 \Rightarrow a_2 = -\frac{\lambda}{2} a_0 \quad \text{—(2)}$$

$$6a_3 - 2a_1 + \lambda a_1 = 0 \quad \text{—(3)}$$

$$a_{k+2} = \frac{k^2+k-\lambda}{(k+1)(k+2)} \quad \text{—(4)}$$

We thus set

$$a_2 = -\frac{\lambda}{2} a_0$$

$$a_4 = \frac{6-\lambda}{4\cdot 3} a_2 = -\frac{\lambda(6-\lambda)}{4!} a_0$$

$$a_6 = \frac{20-\lambda}{6\cdot 5} a_4 = -\frac{\lambda(6-\lambda)(20-\lambda)}{6!} a_0$$

— — —

$$a_{2n} = \frac{2n(2n+1)-\lambda}{(2n+2)(2n+1)} a_{2n-2}$$

$$a_3 = \frac{2-\lambda}{6} a_1$$

$$a_5 = \frac{12-\lambda}{5\cdot 4} a_3 = \frac{(2-\lambda)(12-\lambda)}{5!} a_1$$

$$a_7 = \frac{30-\lambda}{7\cdot 6} a_5 = \frac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!} a_1$$

— —

$$a_{2n+1} = \frac{(2n+1)(2n+2)}{(2n+2)(2n+3)} a_{2n-1}$$

We can therefore write

$$y(x) = a_0\left(1 - \frac{\lambda}{2}x^2 - \frac{\lambda(6-\lambda)}{4!}x^4 - \frac{\lambda(6-\lambda)(20-\lambda)}{6!}x^6 - \cdots\right)$$

$$+ a_1\left(x + \frac{2-\lambda}{3!}x^3 + \frac{(2-\lambda)(12-\lambda)}{5!}x^5 + \cdots\right.$$

The two linearly independent solutions are

$$y_e(x) = 1 - \frac{\lambda}{2}x^2 - \frac{\lambda(6-\lambda)}{4!}x^4 - \frac{\lambda(6-\lambda)(20-\lambda)}{6!}x^6 - \cdots$$

$$y_o(x) = x + \frac{2-\lambda}{3!}x^3 + \frac{(2-\lambda)(12-\lambda)}{5!}x^5 + \cdots$$

Legendre Polynomials:

From above we observe

$\lambda = 0$, $y(x) = a_0$ (y_e gives this term)

$\lambda = 2$, with $a_0 = 0$, $y(x) = a_1(x)$

$\lambda = 6$ with $a_1 = 0$, $y(x) = a_0(1 - 3x^2)$

$\lambda = 12$ with $a_0 = 0$, $y(x) = a_1(x - \frac{5}{3}x^3)$

$\lambda = 20$ with $a_1 = 0$, $y(x) = a_0(1 - 10x^2 + \frac{35}{3}x^4)$

The values of λ for which we get polynomial solutions are $\lambda = n(n+1)$, $n = 0, 1, 2, 3, \ldots$

If we choose a_0, a_1 in such a way that $P_n(1) = 1$, we can write the Legendre Polynomials

$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x), \quad P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x) \quad \cdots$

Thus $P_n(x)$ are solutions of

$((1-x^2)y')' + n(n+1)y = 0, \quad -1<x<1$

n a non negative integer;

Orthogonality Relation : $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0, \; m \neq n.$

Fourier-Legendre Series :

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

where $a_n = \dfrac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} P_n^2(x)\, dx}$

General Formula

$$P_n(x) = \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!}{2^n\, k!\, (n-2k)!\, (n-k)!} x^{n-2k}$$

where $m = \frac{n-1}{2}$ or $\frac{n}{2}$ whichever is an integer

In other words $m = \left[\frac{n}{2}\right]$ (bracket function)

$[x]$ = largest integer $\leq x$.