Chapter 3

Orthonormal Sets and Fourier Series

In this chapter we shall formally present the series expansion of a 'reasonably well behaved' function $f(x)$ which appeared towards the end of solution by separation of variables. We shall first give a general expansion in terms of eigenfunctions of problems of Sturm - Liouville type.

Inner Product : We consider the so-called L_2 space over interval $[a,b]$ denoted by $L_2[a,b]$. This space consists of square integrable functions over $[a,b]$. The inner product $\langle \, , \, \rangle$ in $L_2[a,b]$ is defined as

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx.$$

The ~~norms~~ norm of a function $f(x)$ is given by

$$\|f\| = \langle f, f\rangle^{1/2} = \left\{ \int_a^b f^2(x)\, dx \right\}^{1/2}.$$

Orthogonal Functions Two functions f and g are orthogonal in $L_2[a,b]$ if

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, g(x)\, dx = 0.$$

A set of functions in $L_2[a, b]$ ~~is~~ ~~a~~ $\{\phi_1(x), \phi_2(x), \dots\}$ is said to be orthogonal set if

$$\langle \phi_i(x), \phi_j(x) \rangle = \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0 \quad i \neq j.$$

<u>Orthonormal Set</u>: The set $\{\phi_1(x), \phi_2(x) \dots\}$ is called an orthonormal set in $L_2[a, b]$ if

(1) $\langle \phi_i(x), \phi_j(x) \rangle = 0 \quad i \neq j$

(2) $\|\phi_i\|^2 = \langle \phi_i, \phi_i \rangle = 1$

In short

$$\langle \phi_i(x), \phi_j(x) \rangle = \int_a^b \phi_i(x)\, \phi_j(x)\, dx$$

$$= \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

δ_{ij} is Kronecker delta.

Examples

(1) $\phi_n(x) = \left\{ \sqrt{\frac{2}{\pi}} \sin nx \, , \ n = 1, 2, \dots \right\}$ is an orthonormal set in $[0, \pi]$.

We note that $\displaystyle\int_0^\pi \sin mx \sin nx \, dx = \begin{cases} 0 \, , & m \neq n \\ \frac{\pi}{2} \, , & m = n \end{cases}$

Hence

(1) $\displaystyle \langle \phi_m , \phi_n \rangle = \frac{2}{\pi} \int_0^\pi \sin mx \sin nx \, dx = 0$

(2) $\displaystyle \|\phi_n\|^2 = \langle \phi_n , \phi_n \rangle = \frac{2}{\pi} \int_0^\pi \sin^2 n x \, dx$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} = 1.$$

Hence the result.

(2) $\{\phi_n(x)\}$ given by

$$\phi_0(x) = \frac{1}{\sqrt{\pi}} \, , \quad \phi_n(x) = \sqrt{\frac{2}{\pi}} \cos nx$$
$$n = 1, 2, 3, -$$

is an orthonormal set in $[0, \pi]$.

① $\displaystyle \langle \phi_0(x) , \phi_n(x) \rangle = \frac{\sqrt{2}}{\pi} \int_0^\pi \cos nx \, dx$

$$= \frac{\sqrt{2}}{\pi} \frac{1}{n} \left[\sin nx \right]_0^\pi = 0$$

$$\langle \phi_m(x) , \phi_n(x) \rangle = \frac{2}{\pi} \int_0^\pi \cos mx \cos nx \, dx$$

$$= 0 \quad \text{for } m \neq n.$$

$$\|\phi_0(x)\|^2 = \langle \phi_0(x), \phi_0(x) \rangle = \frac{1}{\pi}\int_0^{\pi} dx$$

$$= \frac{1}{\pi}[x]_0^{\pi} = 1.$$

$$\|\phi_n(x)\|^2 = \langle \phi_n(x), \phi_n(x) \rangle = \frac{2}{\pi}\int_0^{\pi} \cos^2 nx \, dx$$

$$= \frac{\cancel{2}}{\pi} \cdot \frac{1}{\cancel{2}} \int_0^{\pi} (1 + \cos 2nx)\, dx = 1.$$

(3) In $[-\pi, \pi]$, the following form an orthonormal set

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \phi_{2n-1}(x) = \frac{1}{\sqrt{\pi}} \cos nx$$

$$\phi_{2n}(x) = \frac{1}{\sqrt{\pi}} \sin nx.$$

The two requirements can be verified easily as in examples (1) and (2).

<u>Eigenfunction Expansions, Generalized Fourier Series:</u>

Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be an orthonormal set in interval $[a, b]$. Given a function $f(x)$ we may write

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x). \qquad (1)$$

To determine, α_n, we may multiply "scalarly" by $\phi_m(x)$ as follows

$$\langle f(x), \phi_m(x) \rangle = \langle \sum_{n=1}^{\infty} \alpha_n \phi_n(x), \phi_m(x) \rangle$$

which means,

$$\int_a^b f(x)\,\phi_m(x)\,dx = \int_a^b \sum_{n=1}^{\infty} \alpha_n \phi_n(x)\,\phi_m(x)\,dx$$

Assuming, at least formally, we can perform the integration term by term, we get

$$\int_a^b f(x)\,\phi_m(x)\,dx = \sum_{n=1}^{\infty} \alpha_n \int_a^b \phi_n(x)\,\phi_m(x)\,dx$$

$$= \sum_{n=1}^{\infty} \alpha_n \delta_{mn} = \alpha_m$$

which enables us to write

$$\alpha_n = \int_a^b f(x)\,\phi_n(x)\,dx. \qquad (2)$$

① and ② together give the so-called generalized Fourier Series.

In the next section, we shall introduce three such series using appropriate orthogonal sets in interval $[0, \pi]$ or $[-\pi, \pi]$.

Fourier Cosine Series

Let $f(x)$ be defined in $[0, \pi]$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

$n = 0, 1, 2, 3, \ldots$

is the Fourier cosine series in $[0, \pi]$.

Fourier Sine Series:

Let $f(x)$ be defined in $[0, \pi]$. Then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$n = 1, 2, 3, \ldots$

is the Fourier Sine Series.

Fourier Series

Let $f(x)$ be defined in $[-\pi, \pi]$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$; $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$n = 0, 1, 2, 3, \ldots$ $\qquad$ $n = 1, 2, 3, \ldots$

Remarks : ① The above series would give value of $f(x)$ which assumes $f(x)$ to be periodic in $[-\pi, \pi]$.

② If $f(x)$ were an even function, i.e.

$$f(-x) = f(x), \qquad -\pi < x < \pi$$

Then the Fourier series will become the Fourier cosine series in $0 < x < \pi$.

If $f(x)$ were an odd function, i.e.

$$f(-x) = -f(x), \qquad -\pi < x < \pi,$$ then the Fourier series will reduce to the Fourier sine series in $0 < x < \pi$.

③ Fourier sine and Fourier cosine series are sometimes known as half-range Fourier series

(4) If we want to write a series in $[0, l]$ or $[-l, l]$ then we may use

$\cos \frac{n\pi x}{l}$, $\sin \frac{n\pi x}{l}$ instead of $\cos nx$, $\sin nx$

and range of integration being $[0, l]$ instead of $[0, \pi]$ and $[-l, l]$ instead of $[-\pi, \pi]$.

Example:

Write Fourier cosine, Fourier sine and Fourier series (in $[0, \pi]$ or $[-\pi, \pi]$ whichever is applicable) for

① $f(x) = e^{x}$

② $f(x) = x$.