Laplace Transform

We define the Laplace transform $F(s)$ of a function $f(x)$, $x>0$ as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty f(x)\, e^{-sx}\, dx , \qquad (1)$$

the inversion being given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{sx}\, ds \qquad (2)$$

It is clear from (1) that for $F(s)$ to exist, we require $f(x)$ to satisfy $\displaystyle\int_0^\infty e^{-cx} |f(x)|\, dx < \infty$

where $Re(s) = c$ for some $c>0$. Such functions $f(x)$ are said to be of <u>exponential order</u>.

The integral in (2) is generally evaluated using the contour integration. The contour from $c-i\infty$ to $c+i\infty$ is referred to as Bromwich contour and c is chosen such that all singularities lie on the left of line from $c-i\infty$ to $c+i\infty$.

We have seen some examples in which integral in (2) was evaluated.

One may easily verify that Laplace transform defined in (1) is linear. Some properties are given below.

(t - independent variable has been used)

- $\mathcal{L}\{y'\} = s\, Y(s) - y(0)$
- $\mathcal{L}\{y''\} = s^2 Y(s) - s\, y(0) - y'(0)$
- $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
- $\mathcal{L}\{H(t-a)\} = \dfrac{e^{-as}}{s}$, Here $H(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a. \end{cases}$

- $\mathcal{L}\{f(t-a)\, H(t-a)\} = e^{-as} F(s)$
- $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s).$
- (Convolution Property)

$$\mathcal{L}\{f * g\} = \mathcal{L}\left\{\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right\}$$

$$= F(s)\, G(s)$$

$$\Updownarrow$$

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

- $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$

Convolution Theorem : $f*g = \int_0^t f(t-u)g(u)\,du$

$$\mathcal{L}\left[f*g\right] = F(s)\,G(s).$$

Consider $F(s)\,G(s) = F(s)\int_0^\infty e^{-st} g(t)\,dt$

$$= \int_0^\infty F(s)\,e^{-st} g(t)\,dt$$

Change dummy variable to u

$$F(s)\,G(s) = \int_0^\infty F(s)\,e^{-su} g(u)\,du \quad -(1)$$

Now $F(s)\,e^{-su} = \mathcal{L}\{f(t-u)\,H(t-u)\}$

$$(1) \Rightarrow F(s)G(s) = \int_0^\infty \mathcal{L}\{f(t-u)\,H(t-u)\}\,g(u)\,du$$

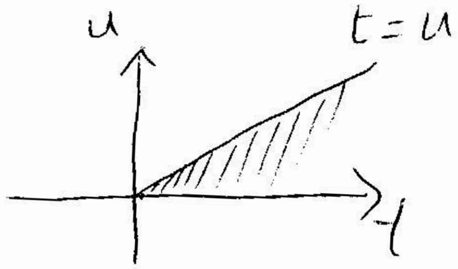
$$= \int_0^\infty \left[\int_0^\infty f(t-u)\,H(t-u)\,e^{-st}\,dt\right] g(u)\,du$$

$$= \int_0^\infty \int_u^\infty e^{-st} g(u)\,f(t-u)\,dt\,du$$

Range

$t = u$ to $t = \infty$

$u = 0$ to ∞



In reverse order : $u = 0$ to $u = t$

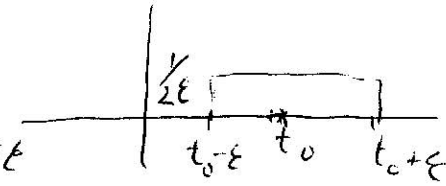
$t = 0$ to $t = \infty$

$$F(s)\,G(s) = \int_0^\infty e^{-st} \underbrace{\int_0^t g(u)\,f(x-u)\,du}\,dt$$

$$= \mathcal{L}\,(f*g).$$

L.T. of Delta function.

Consider $f_{t_0}(t)$ as shown



$$f_{t_0}(t) = \frac{1}{2\varepsilon} \qquad t_0 - \varepsilon < t < t_0 + \varepsilon$$

$$= 0 \qquad \text{otherwise}$$

$$\mathcal{L}\{f_{t_0}(t)\} = \int_0^\infty f_{t_0}(t)\, e^{-st}\, dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \frac{1}{2\varepsilon} e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-2\varepsilon s}\right]_{t_0-\varepsilon}^{t_0+\varepsilon} = \frac{e^{-s(t_0-\varepsilon)} - e^{-s(t_0+\varepsilon)}}{2\varepsilon s}$$

$$= \frac{e^{-st_0}}{s} \frac{e^{s\varepsilon} - e^{-s\varepsilon}}{2\varepsilon}$$

Take limit as $\varepsilon \to 0$, $\quad f_{t_0}(t) \to \delta(t-t_0)$

$$\mathcal{L}\{\delta(t-t_0)\} = \frac{e^{-st_0}}{s} \lim_{\varepsilon \to 0} \frac{e^{s\varepsilon} - e^{-s\varepsilon}}{2\varepsilon}$$

$$= \frac{e^{-st_0}}{\cancel{s}} \cdot \cancel{s} \lim_{\varepsilon \to 0} \frac{e^{s\varepsilon} + e^{-s\varepsilon}}{2} \quad \text{(L'Hopital rule)}$$

$$= e^{-st_0}$$