Solution of PDE / BVP :

Ex(1) Consider string of length 1, rigidly fixed at the two ends. Equation of motion :

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < 1,\ t > 0$$

$$u(0,t) = 0, \qquad u(1,t) = 0$$

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = \sin \pi x, \quad 0 < x < 1.$$

Taking Laplace transform in t

$$\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = s^2 U(x,s) - s\, u(x,0) - u_t(x,0)$$

$$= s^2 U(x,s) - \sin \pi x$$

$$\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2}\left\{\mathcal{L}\, u(x,t)\right\} = \frac{d^2 U}{dx^2}$$

$$\mathcal{L}\{u(0,t) = 0\} \Rightarrow U(0,s) = 0$$

$$\mathcal{L}\{u(1,t) = 0\} \Rightarrow U(1,s) = 0$$

PDE $\Rightarrow$: $\dfrac{d^2 U}{dx^2} = s^2 U - \sin \pi x$

or $\left.\begin{array}{l} d^2 U - s^2 U = -\sin \pi x \\ U(0,s) = 0, \quad U(1,s) = 0 \end{array}\right\}$

This has solution (class work)

$$U(x,s) = \frac{1}{s^2 + \pi^2} \sin \pi x$$

Inversion gives

$$u(x,t) = \frac{1}{\pi} \sin \pi x \sin \pi t$$

Example 2. Consider a semi infinite string, $x>0$

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x>0,\ t>0$$

$$u(0,t) = f(t), \quad \lim_{x\to\infty} u(x,t) = 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x>0$$

Taking Laplace transform in t and using initial conditions, PDE $\Rightarrow$

$$\frac{d^2 U}{dx^2} = \frac{s^2 U}{c^2} \quad \text{———} \ *$$

$$\mathcal{L}\{u(0,t) = f(t)\} \Rightarrow U(0,s) = F(s)$$

$$\mathcal{L}\left\{\lim_{x\to\infty} u(x,t) = 0\right\} \Rightarrow \lim_{x\to\infty} U(x,s) = 0$$

The general solution of $*$ is

$$U(x,s) = C_1 e^{-\frac{s}{c}x} + C_2 e^{+\frac{s}{c}x}$$

$$\lim_{x\to\infty} U(x,s) = 0 \Rightarrow C_2 = 0$$

$$U(0,s) = F(s) \Rightarrow C_1 = F(s)$$

Hence $U(x,s) = F(s)\, e^{-\frac{x}{c}s}$

Inverse Laplace transform gives

$$u(x,t) = f\left(t-\frac{x}{c}\right) \mathcal{U}\left(t-\frac{x}{c}\right)$$

(Second Shifting Theorem).

Example 3 **Telegraph Equation**

If v is voltage, i current, L inductance and C denotes capicitance, then

$$\frac{\partial^2 v}{\partial x^2} = LC\,\frac{\partial^2 v}{\partial t^2}$$

At $t=0$, line is dead so

$$v(x,0)=0, \quad \frac{\partial v}{\partial t}(x,0)=0$$

At $x=0$, $v(0,t) = f(t)$

At $x=l$, $\frac{\partial v}{\partial x}(l,t) = 0$

We can solve it using Laplace transform and when $l \to \infty$, the solution can be written as

$$v(x,t) = \begin{cases} f\left(t-\frac{x}{c}\right), & t > \frac{x}{c} \\ 0, & t < \frac{x}{c} \end{cases}$$

(class work)